NOTES ON TREES

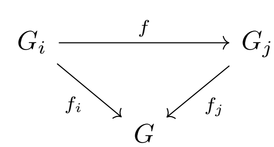
ERIC ALBERS

Contents

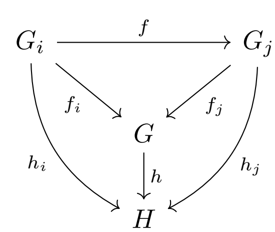
These notes are based off of a reading course out of Serre's *Trees*.

1. Amalgamated Free Products

The main construction at the level of groups which will be studied extensively throughout these notes is the *amalgamated free product.* To discuss this, let us first recall the notion of the categorical (direct) limit from which we will obtain the amalgamated free product as a specific case. Let $(G_i)_{i\in I}$ be an indexed family of groups, $F_{i,j} \subseteq \mathrm{Hom}_{\mathbf{Grp}}(G_i, G_j)$. A group G together with homomorphisms $f_i : G_i \to G$ such that



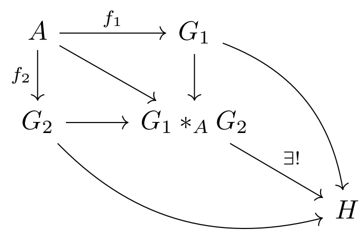
commutes for any $f \in F_{i,j}$ is said to be the direct limit $\lim_{\to} G_i$ if it satisfies the following universal property: For any other group H with homomorphisms $h_i : G_i \to H$ making the obvious diagram commute, there exists a unique $h : G \to H$ such that



commutes for every $i, j, f \in F_{i,j}$.

With the tools of group presentations the existence of limits is quite easy. For the generating set of G take the generators of all the G_i. For relations first impose all of the relations that held in each of the G_i, and add on to to this the relation $x = y$ for $x \in G_i, y \in G_j$ if for some $f \in F_{i,j}$, we have $y = f(x)$.

As an example of this construction, let A, G_1, G_2 be groups and suppose $f_1 : A \to G_1, f_2 : A \to G_2$ are group homomorphisms. Combining the appropriate diagrams for the direct limit, which in this case we will denote by $G_1 *_A G_2$, we see it satisfies the universal property of the pushout:



If our groups have the presentations,

$$A = \langle \mathscr{S}_A | \mathscr{R}_A \rangle, \quad , G_1 = \langle \mathscr{S}_1 | \mathscr{R}_1 \rangle, \quad \langle \mathscr{S}_2 | \mathscr{R}_2 \rangle,$$

then as our description,

$$G_1 *_A G_2 = \langle \mathscr{S}_A, \mathscr{S}_1, \mathscr{S}_2 | \mathscr{R}_A, \mathscr{R}_1, \mathscr{R}_2, a = f_1(a) = f_2(a) \text{ for all } a \in A \rangle,$$

which of course simplifies to the pushout in groups given by the presentation,

$$G_1 *_A G_2 = \langle \mathscr{S}_1, \mathscr{S} | \mathscr{R}_1, \mathscr{R}_2, f_1(a) = f_2(a) \text{ for all } a \in A \rangle.$$

We will know generalize the above to larger families of groups while insisting that the group homomorphisms be injective, so that more can be said about the structure of the resulting amalgam. Formally, let A be a group and $(G_i)_{i \in I}$ be a famnily of groups. For each i, let $f_i : A \to G_i$ be an injective group homomorphism allowing use to think of A as a subgroup of each G_i. We will denote the corresponding direct limit to these groups and maps as $*_A G_i$.

With just this extra structure of injectivity of the G_i, we can obtain a canonical decomposition of elements in $*_A G_i$ which before proving we must first discuss the notion of a *reduced word*. For each $i \in I$ let S_i be a set of right coset representatives of elements of $A \backslash G_i$ which we insist contains $1 \in S_i$. Then the map $A \times (G_i \backslash \{1\}) \to G_i \setminus A$ given by $(a, s) \mapsto as$ is bijective. If $\underline{i} = (i_1, \dots, i_n)$ is a finite sequence of elements of I such that $i_m \neq i_{m+1}$ for all m, a reduced word of type $\underline{i}$ is a family

$$m = (a; s_1, \dots, s_n),$$

where $a \in A, s_1 \in G_{i_1}, \dots, s_n \in G_{i_n}$. Before stating and proving the structure theorem for amalgams of this form, fix $G = *_A G_i$ and fix the canonical homomorphisms $f : A \to G, f_i : G_i \to G$

Theorem 1.1. *For any $g \in G$, there exists a finite sequence $\underline{i} = (i_1, \dots, i_n)$ of elements of I such that $i_m \neq i_{m+1}$ for each m and a reduced word $m = (a; s_1, \dots, s_n)$ of type $\underline{i}$ such that*

$$g = f(a) f_{i_1}(a_1) \dots f_{i_n}(s_n).$$

Furthermore both $\underline{i}$ and m are unique.

Note before we prove this theorem that this result implies that f and each of the f_i are all injective, which is certainly not a priori clear from the construction given by the group presentation.

Proof. For a given vector $\underline{i}$ we let $X_{\underline{i}}$ denote the collection of all reduced words of type $\underline{i}$ and let X denote the disjoint union of the $X_{\underline{i}}$ over all $\underline{i}$. Our goal will be to construct a group action of G on X, but by the construction of the direct limit, it suffices to construct an action of each of the G_i on X for which the induced action of A on X viewing $A \leq G_i$ does not depend on our choice of i.

To this end, for $i \in I$, let Y_i be the set of reduced words of the form $(1; s_1, \dots, s_n)$ where $i_1 \neq i$. We then have maps to X given by the following assignments,

$$\begin{aligned} A \times Y_i &\to X \\ (a, (1; s_1, \dots, s_n)) &\mapsto (a; s_1, \dots, s_n) \\ A \times (S_i \setminus \{1\}) \times Y_i &\to X \\ ((a, s), (1; s_1, \dots, s_n)) &\mapsto (a; s, s_1, \dots, s_n) \end{aligned}$$

which yield a clear bijection $(A \times Y_i) \cup (A \times (S_i \setminus \{1\}) \times Y_i) \to X$. But note since we can identity $A \cup A \times (S_i \setminus \{1\})$ with G_i, this map should be thought of as a bijection $\theta : G_i \times Y_i \to X$. Explicitly for $a \in A \leq G_i$, $\theta(a, (1; s_1, \dots, s_n)) = (a; s_1, \dots, s_n)$ and for $g \in G_i \setminus A$ such that $g = as$ for $a \in A, s \in S_i$, $\theta(g, (1; s_1, \dots, s_n)) = (a; s, s_1, \dots, s_n)$. Thus viewing X as the set $G_i \times Y$, G_i acts in the obvious way,

$$g'.(g, y) = (g'g, y).$$

The restriction of this action to A is easily to seen to be given by

$$a'.(a; s_1, \dots, s_n) = (a'a; s_1, \dots, s_n),$$

which is evidently independent of i.

As mentioned this now constructs an action of all of G on X and a simple computation checks that if

$$m = (a; s_1, \dots, s_n),$$

is a reduced word and $g = f(a)f_{i_1}(s_1) \dots f_{i_n}(s_n) \in G$ then the result of the action of g on $(1;)$ is precisely the word m. Now, let $\alpha : G \to X$ be the map given by $\alpha(g) = g.(1;)$ and $\beta : X \to G$ be given by

$$\beta(a; s_1, \dots, s_n) = f(a)f(s_1) \dots f(s_n).$$

Our previous remark yields that $\alpha \circ \beta = \mathrm{Id}_X$ and so β is injective, i.e. the decomposition of an element into the desired form is unique. To show each element can be decomposed into such a form, note since $G_i X \subseteq X$ for each X, we have that $GX \subseteq X$ and so upon viewing G as its image $G.(1;)$ we get $G \subseteq X$ and so $G = X$. $\square$

With this eminently useful theorem on the structure of amalgamated free products proven we now discuss some consequences. First allow us to fix some notation. We will fix $G = *_A G_i$ throughout. For an element $g \in G$, we say its type is $\underline{i} = (i_1, \dots, i_n)$ where $\underline{i}$ corresponds to the decomposition of g as in the theorem. We have that $\underline{i} = \varnothing$ if and only if $g \in A$. The *length* of g, denoted $\ell(g)$, is simply n as in the sequence $\underline{i}$ above. Trivially $\ell(g) \leq 1$ if and only if g belongs to one of the G_i. We say g is *cyclically reduced* if g is of type $\underline{i} = (i_1, \dots, i_n)$ where $i_1 \neq i_n$.

Proposition 1.2.

a) *Every $g \in G$ is conjugate to either a cyclically reduced element or an element of one of the G_i.*
b) *Cyclically reduced elements have infinite order.*

Proof. For a we proceed by induction on $\ell(g)$ with the case of $\ell(g) = 1$ being clear. Thus if $\ell(g) \geq 2$ and g is not cyclically reduced of type $\underline{i} = (i_1, \ldots, i_n)$, we may write

$$g = g_1 \ldots g_n,$$

where each $g_i \in G_{i_1} \setminus A$. Since $i_1 = i_n$, upon conjugating by g_1 we see

$$g_1^{-1} g g_1 = g_2 \ldots g_{n-1}(g_n g_1),$$

which has length at most $n-1$. Thus by induction, $g_1^{-1} g g_1$ is conjugate to either a cylically reduced element or an element of some G_i and so the same holds true for g.

For b simply note if g is of type $(i_1, \ldots, i_n)$ and is cyclically reduced then g^2 is of type $(i_1, \ldots, i_n)$. In general, g^k has length kn and so g is of infinite order. □

This proposition has some immediate corollaries.

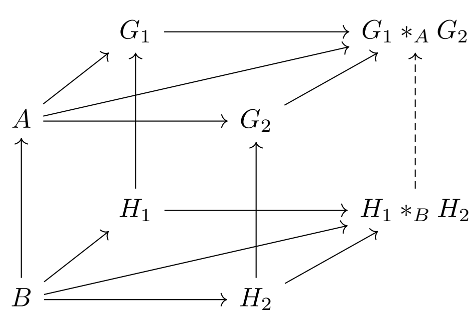
Corollary 1.3. *All finite order elements of G are conjugate to an element of one of the G_i. Consequently G is torsion-free if the G_i are.*

It is remarked in Serre's book that a stronger statement is true: All finite subgroups of $*_A G_i$ are conjugate to a subgroup of one of the G_i.

Proposition 1.4. *For all $i \in I$ let $H_i \leq G_i$ and suppose that $B = H_i \cap A$ is independent of i. Then the homomoprhism $*_B H_i \to *_A G_i$ induced by the inclusion maps $H_i \to G_i$ is injective.*

Proof. As in our structure theorem, let T_i denote a set of right coset representatives for each $B \backslash H_i$ which, since $B = A \cap H_i$, can be extended to a set of right coset representatives of each $A \backslash G_i$. Then the unique decompositions with respect to the T_i correspond to unique decompositions in $*_A G_i$ with respect to the S_i, giving the proposition. □

Thus for any collection of subgroups $H_i \leq G_i$ which all intersect A trivially, we obtain a copy of the free product $*H_i$ as a subgroup of $*_A G_i$. In the case of pushouts this theorem can be thought of as the following diagram, where existence of the map $H_1 *_B H_2 \to G_1 *_A G_2$ exists and is unique by the universal property of the pushout.



Next a general fact about free products.

Proposition 1.5. *Let A, B be groups and R be the kernel of the canonical homomorphism $A * B \to A \times B$. The group R is a free group with free basis the set X containing all elements of the form $[a, b] = a^{-1}b^{-1}ab$ for $a \in A, b \in B$.*

Proof. This group homomorphism is obviously surjective and $X \subseteq R$. Moreover using S to denote the subgroup generated by X, for any $a, a' \in A, b \in B$,

$$a'^{-1}[a, b]a = a'^{-1}(a^{-1}b^{-1}ab)a' = [aa', b][a', b]^{-1} \in S,$$

and similarly for any $b' \in B$ allowing us to conclude normality of S. Now $A * B/S$ clearly has the presenation given by

$$\langle \mathscr{S}_A, \mathscr{S}_B | \mathscr{R}_A, \mathscr{R}_B, ab = ba \text{ for any } a \in A, b \in B\rangle = A \times B,$$

from which we conclude that $R = S$.

It remains to show that X is a free subset. To this end, let $[a_1, b_1], \dots, [a_n, b_n] \in X$ and suppose

$$g = [a_1, b_1]^{\epsilon_1} \dots [a_n, b_n]^{\epsilon_n},$$

where each $\epsilon_i \in \{\pm 1\}$ and it does not simultaneoulsy hold that $a_i = a_{i+1}, b_i = b_{i+1}, \epsilon_i = -\epsilon_{i+1}$. We aim to show that $g \neq 1$. Indeed we will show that $\ell(g) \geq n+3$ and if $\epsilon_n = 1$ (resp. -1) then the canonical decomposition for g ends in $a_n b_n$ (resp. $b_n a_n$). We show this via induction on n with the case of $n = 1$ being trivial. Assume without loss of generality that $\epsilon_{n-1} = 1$. Then by induction,

$$g' = [a_1, b_1]^{\epsilon_1} \dots [a_{n-1}, b_{n-1}]^{\epsilon_{n-1}},$$

has a reduced decomposition of the form,

$$g' = s_1 \dots s_p a_{n-1} b_{n-1},$$

where $p \geq n$. Hence if $\epsilon_n = 1$,

$$g = s_1 \dots s_p a_{n-1} b_{n-1} a_n^{-1} b_n^{-1} a_n b_n,$$

which is a reduced decomposition of length at least $n+6$ ending in $a_n b_n$. If instead $\epsilon_n = -1$,

$$g = s_1 \dots s_p a_{n-1}(b_{n-1} b_n^{-1}) a_n^{-1} b_n a_n.$$

If $b_{n-1} b_n^{-1} \neq 1$ then this is a reduced decomposition of length at least $n+5$ terminating in $b_n a_n$. If $b_{n-1} b_n^{-1} = 1$ it cannot hold that $a_{n-1} a_n^{-1} = 1$ and so

$$g = s_1 \dots s_p (a_{n-1} a_n^{-1}) b_n a_n,$$

which has length at least $n+3$ terminating in $b_n a_n$ as claimed. □

Let us now move on to discuss some classical constructions using the amalgamated free product, the first of which being what is known as HNN extensions.

Theorem 1.6. *Let G be a group, $A \leq G$ and $\theta : A \to G$ an injective group homomorphism. Then there is a group G' containing G and an element $s \in G'$ such that for all $a \in A$, $\theta(a) = sas^{-1}$. Moreover if G is countable, finitely generated, or torsion-free, G can be chosen to have the same property.*

Proof. Let $A_n = A, G_n = G$ for all $n \in \mathbb{Z}$. Let H be the group obtained by amalgamting the A_n and G_n with respect to the homomorphisms $\theta : A_n \to G_n$ and the inclusion map $A_n \to G_{n+1}$. Explictly H is the $\mathbb{Z}$-fold free product of G together with the added relation that for any $a \in A \leq G_{n+1}$ we have $a = \theta(a) \in G_n$. Now let $u_n : G_n \to G_{n+1}$ be the canonical isomorphism. These u_n define a shifting automorphism $u : H \to H$ explictly given by for $g \in G_n$, $u(g)$ is the copy of g

inside G_{n+1}. Identifying G with G_0 then for $a \in A \leq G_0$ we have $u(a) \in G_1$ is identified with $\theta(a)$ in G_0. Thus u is an automorphism of H extending θ. Now let S be an infinite cyclic group generated by s. S acts on H via $s.h = u(h)$. We then define G' to be the external semi-direct product, $G' := H \rtimes S$. Explicitly the group structure on G' is defined via,

$$(h_1, s^{k_1})(h_2, s^{k_2}) = (h_1 u^{k_1}(h_2), s^{k_1+k_2}).$$

We check that G' together with s has the desired property. Indeed for any $a \in A$,

$$\begin{aligned}(1, s)(a, 1)(1, s^{-1}) &= (u(a), s)(1, s^{-1}),\\ &= (u(a), 1),\\ &= (\theta(a), 1)\end{aligned}$$

as desired. □

Morally this theorem should be thought of as constructing from a group G which contains isomorphic subgroups H, K, a group G' containing G for which H and K are conjugate in G'. Explicitly a presentation for this group is given by

$$\langle \mathscr{S}_G, s | \mathscr{R}_G, sas^{-1} = \theta(a) \text{ for any } a \in A\rangle.$$

While it may appear that our group G' as constructed is generated by infinitely many copies of G, in the semi-direct product conjugation by s allows one to move between copies of the group G.

Corollary 1.7. *Every group G can be embedded in a group K such that all elements in K of the same order are conjugate. Moreover if G is countable or torsion-free, K can be chosen to have the same property.*

Proof. Let $x, y \in G$ be elements of the same order. By Theorem 1.6 we can construct a larger group G_{xy} such that x and y are conjugate in G_{xy}. Continuing this process iteratively and passing to the direct limit (which is really just a union in this context) we obtain a group $E(G)$ such that all elements of G of the same order are conjugate. Of course not all elements of $E(G)$ of the same order are necessarily conjugate but upon taking the union over $E(E(\dots(E(G))))$ we obtain the desired group. □

Note that this corollary says if G is a torsion-free group it can be embedded into a simply group K.

For our next construction using amalgams we discuss the so-called Higman group.

Theorem 1.8. *Let G be the group generated by the elements x_1, x_2, x_3, x_4 together with the relations,*

$$x_2 x_1 x_2^{-1} = x_1^2, \quad x_3 x_2 x_3^{-1} = x_2^2, \quad x_4 x_3 x_4^{-1} = x_3^2, \quad x_1 x_4 x_1^{-1} = x_4^2.$$

Then the only finite index subgroup of G is G itself and G is infinite.

Proof. For the first claim, since any finite index group contains a finite index normal subgroup, it suffices to show that G does not have any non-trivial finite quotients. Supposing by way of contradiction $\overline{G}$ was such a quotient, let n_i denote the order of x_i in the quotient for $i = 1, \dots, 4$. Since $\overline{G}$ is non-trivial at least one of $n_i > 1$ and thus we let p be the smallest prime divisor of any of the n_i. Without loss of generality we assume $p | n_1$. A simply induction argument shows for any $n \in \mathbb{N}$,

$$x_1^{2^n} = x_2^n x_1 x_2^{-n}.$$

Thus for $n = n_2$ we get

$$x_1^{2^{n_2}} = x_2^{n_2} x_1 x_2^{-n_2} = x_1 \text{ in } \overline{G}.$$

Hence $2^{n_2} \equiv 1 \pmod{n}_1$ and thus $2^{n_2} \equiv 1 \pmod{p}$. In other words, if N denotes the order of [2] in $(\mathbb{Z}/p\mathbb{Z})^\times$, we have that $N|n_2$. Since $N \leq p-1$ we see that any prime factor p' of N is smaller than p and a prime factor of n_2, contradicting the minimality of p. Thus G has no finite quotients.

To show that G is infinite we will show it is an amalgam which is evidently infinite. Let G_{12} denote the group generated by x_1, x_2 and the relation $x_2 x_1 x_2^{-1} = x_1^2$. One can explicitly check that both x_1 and x_2 have infinite order in G_{12}, so let G_1, G_2 be the respective copies of $\mathbb{Z}$ they generate and also note $G_1 \cap G_2 = \{1\}$. Define similarly the groups G_{23}, G_{34}, and G_{41}. We can amalgamate G_{12}, G_{23} over their shared subgroup G_2 to get the group,

$$G_{123} = G_{12} *_{G_2} G_{23} = \langle x_1, x_2, x_3 | x_2 x_1 x_2^{-1} = x_1^2, x_3 x_2 x_3^{-1} = x_2^2 \rangle,$$

and similarly,

$$G_{341} = G_{34} *_{G_4} G_{41} = \langle x_3, x_4, x_1 | x_4 x_3 x_4^{-1}, x_1 x_4 x_1^{-1} = x_4^2 \rangle.$$

Applying Proposition 1.4 to to the subgroups $G_1 \leq G_{12}, G_3 \leq G_{12}$ which both intersect G_2 trivially we see that G_{123} contains a copy of F_2, namely $G_1 * G_3$. The same can be said for G_{341} and since

$$G = G_{123} *_{F_2} G_{341},$$

we see that G is infinite since in particular it contains F_2 as a subgroup. □

2. Trees

2.1. **Salient Facts on CW-Complexes.** Since it will be useful to view our trees as CW-Complexes, before developing the introductory theory of trees, we provide some necessary facts on CW-Complexes. Recall that a CW-Complex is a topological space constructed in the following manner: One starts with a discrete set X^0 whose points are the so-called 0-cells. Then one inductively defines the n-skeleton X^n by attaching n-cells e_α^n via attaching maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. More concretely X_n is the quotient of the disjoint union $X^{n-1} \coprod (\bigsqcup_\alpha D_\alpha^n)$ under the equivalence relation $x \sim \varphi_\alpha(x)$ for $x \in \partial D_\alpha^n$. One can stop after finitely many steps, at which point X is set equal to X^n, or if there are n-cells for infinitely many n one defines $X := \bigcup_n X_n$ and gives X the weak topology: i.e. the weakest topology so that the inclusion maps $X_n \hookrightarrow X$ are continuous. Explicitly this topology says that $A \subseteq X$ is open if and only if it intersects each X^n in an open set.

A crucial question in topology is the following: Given a topological space X and a subspace A, if one has a homotopy $h : A \times I \to Y$ and a map $f : X \to Y$ such that $f|_A = h_0$, when does there exist a homotopy on X, $H : X \times I \to Y$ such that $H_0 = f$ and $H_t|_A = h_t$ for all $t \in I$? We will show that such an extension exists in the context where X is a CW-complex and A is a subcomplex. First, an equivalent condition is given by the following proposition.

Proposition 2.1. *A pair (X, A) has the homotopy extension property if and only if $X \times \{0\} \cup A \times X$ is a retraction of the space $X \times I$.*

Proof. For ease of proof we will assume that A is a closed subspace, although we remark that as the proposition states, the results holds in general. If (X, A) has the homotopy extensions property then the identity map on $X \times \{0\} \cup A \times X$ admits an extension φ making the following diagram commute:

$$\begin{array}{ccc} X \times \{0\} \cup A \times I & \xrightarrow{\mathrm{Id}} & X \times \{0\} \cup A \times I \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle \varphi} & \\ X \times I & & \end{array}$$

and consequently φ is the desired retraction.

Conversely if $h : A \times I \to Y$ is a homotopy for which the map $f : X \to Y$ agrees with h_0 on A, we then obtain a continuous map $(f, g) : X \times \{0\} \cup A \times X \to Y$ with the obvious definition. Now, given a retraction $r : X \times I \to X \times \{0\} \cup A \times X$, the composition $(f, g) \circ r$ is the desired homotopy. □

With this equivalent formulation we can show CW-pairs have the homotopy extension property.

Proposition 2.2. *If X is a CW complex and A a subcomplex, then (X, A) has the homotopy extension property.*

Proof. Note that there is a retraction $r : D^n \times I \to D^n \times \{0\} \cup (\partial D^n \times I)$, for example one can take the radial projection from the point $D^n \times \{2\}$ in $D^n \times \mathbb{R}$. Now since the space $X^n \times I$ is obtained by attaching copies of $D^n \times I$ along $D^n \times \{0\} \cup \partial D^n \times I$ we can therefore retract $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$. The desired retraction is then obtained by performing the retraction of the n-skeleton during time $[1/2^{n+1}, 1/2^n]$. □

2.2. **Graphs and Trees.** A *graph* Γ consists of a vertex set $X = \operatorname{vert}\Gamma$ and an edge set $Y = \operatorname{edge}\Gamma$ together with a map $Y \to X \times X$ given by

$$y \mapsto (o(y), t(y)),$$

and a map $Y \to Y$ given by $y \mapsto \overline{y}$ such that for every $y \in Y$ one has,

$$\overline{\overline{y}} = y, \quad \overline{y} \neq y, \quad o(y) = t(\overline{y}).$$

A morphism f of graphs is then simulatenously two maps $f : \operatorname{vert}\Gamma \to \operatorname{vert}\Gamma', f : \operatorname{edge}\Gamma \to \operatorname{edge}\Gamma'$ such that for every $y \in \operatorname{edge}\Gamma$, $f(y)$ is such that

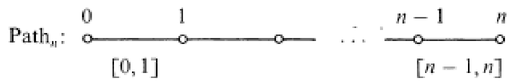
$$o(f(y)) = f(o(y)), \quad t(f(y)) = f(t(y)).$$

One should note that we have chosen our graphs to be undirected by viewing each edge as two edges, one for the possible orientations. An orientation of a graph Γ is simply a subset $Y_+ \subseteq \operatorname{edge}\Gamma$ such that

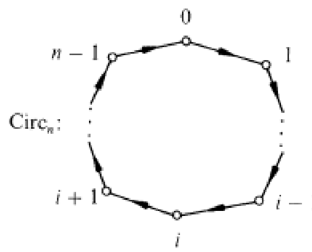
$$Y = Y_+ \sqcup \overline{Y_+}.$$

We now construct a topological space called the realization of Γ which will give a graph the natural one-dimensional CW complex structure associated to the diagram one usually draws for a graph. To this end let Γ be a graph, $X = \operatorname{vert}\Gamma, Y = \operatorname{edge}\Gamma$. Let $T = X \coprod (Y \times [0, 1])$ where we view X, Y with the discrete topology. Let R be the finest equivalence relation such that $(y, t) \equiv (\overline{y}, 1-t), (y, 0) \equiv o(y), (y, 1) \equiv t(y)$. The quotient T/R is then the realization of Γ denoted $\operatorname{real}\Gamma$.

Consider the following graph denoted Path_n.



A *path of length* n in Γ is a morphism of graphs $c : \mathrm{Path}_n \to \Gamma$. A graph is said to be connected if every two vertices are the extremities of a path in Γ. We will write a path as its corresponding sequence of edges $(y_1, \dots, y_n)$ and say that this path is *without backtracking* if for all i one has that $y_i \neq \overline{y_{i+1}}$. We have a similar graph, the circuit of length n which is denoted Circ_n and given by the diagram,



A *circuit of length* n in a graph Γ is a subgraph of Γ which is isomorphic to Circ_n. A circuit of length 1 will be called a *loop*. For one last bit of preliminary terminology we will say a graph Γ such that for any $P, Q \in \mathrm{vert}\,\Gamma$, there is at most one edge with origin P and terminus Q is *combinatorial*.

Let G be a group and $S \subseteq G$. We form the graph $\Gamma = \Gamma(G, S)$ which is the oriented graph with vertex set G, and $(\mathrm{edge}\,\Gamma)_+ = G \times S$ where $o(g, s) = g$ and $t(g, s) = gs$. Note that necessarily G acts freely on vertices and edges.

Proposition 2.3. *Let G be a group, $S \subseteq G$ and $\Gamma = \Gamma(G, S)$. Then*
- *a) Γ is connected if and only if $\langle S \rangle = G$,*
- *b) Γ contains a loop if and only if $1 \in G$,*
- *c) Γ is combinatorial if and only if $S \cap S^{-1} = \varnothing$.*

We now move on to discussing the real objects of interest in these notes: trees. A *tree* is a connected graph with no circuits. A *geodesic* in a tree is a path without backtracking.

Proposition 2.4. *Let Γ be a tree. Then for any $P, Q \in \mathrm{vert}\,\Gamma$ there exists a unique geodesic from P to Q and this geodesic is an injective path.*

Proof. Of course existence follows from connectivity of Γ. Injectivity is also obvious since if any two points on the path agree, the segment of the path between these instances is a circuit in Γ. For uniqueness let $(y_1, \dots, y_n), (w_1, \dots, w_m)$ be two geodesics in Γ connecting P to Q. It must hold that $y_n = w_m$ since otherwise $(y_1, \dots, y_n, \overline{w_m}, \dots, \overline{w_1})$ would be a circuit in Γ. Induction completes the argument. □

We then obtain a metric on the set of verticies of Γ by saying $d(P, Q)$ is precisely the length (counting the edges) of the geodesic from P to Q. Sometimes we will denote this metric by $\ell(P, Q)$.

Fix a vertex P of a tree and for $n \geq 0$ let X_n denote the set of vertices of distance n from P. If $Q \in X_n$, uniqueness of geodesics gives a unique vertex Q' at distance $< n$ from P with Q' adjacent to Q. In particular it must be the vertex $o(y_n)$ where

$(y_1, \ldots, y_n)$ is the geodesic from P to Q. This yields a map $f_n : Q \mapsto Q'$ from $X_n \hookrightarrow X_{n-1}$, yielding the inverse system

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0\{P\}.$$

Moreover given such an inverse system we get a tree by setting $\operatorname{vert}\Gamma = \bigcup_n X_n$ and defining geometric edges via $\{Q, f_n(Q)\}$ for $Q \in X_n$. In particular this shows an equivalence of categories between points trees and inverse systems of sets indexed by $\mathbb{N}$.

For Γ a tree, X' a subset of $X = \operatorname{vert}\Gamma$, then every subtree of Γ containing X' contains the geodesics in Γ connecting points in X' and conversely X' together with the edges making up these geodesics is a subtree. This subtree Γ' is the tree generated by X'. It is an easy exercise to prove that if X' has diameter n, then Γ' also has diameter n.

We claim the realization of any tree is contactible. To see this, fix a point $P \in \operatorname{vert}\Gamma$ and Q a vertex at distance n from P. Then $\Gamma(P,Q) \cong \mathrm{Path}_n$ and Path_n is contractible. Writing

$$\Gamma = \bigcup_Q \Gamma(P,Q),$$

we see that

$$\operatorname{real}\Gamma = \bigcup_Q \operatorname{real}(\Gamma(P,Q),$$

and so $\operatorname{real}\Gamma$ is contractible.

For Γ a graph, $X = \operatorname{vert}\Gamma, Y = \operatorname{edge}\Gamma$, let $P \in X$ and Y_p denote the set of edges y with $P = t(y)$. $|Y_p|$ is said to be the *index* of P. If $n = 0$ then P is *isolated* and if $n = 1$ then P is *terminal.*. Given a terminal vertex P we then form the graph $\Gamma \setminus P$ such that $\operatorname{vert}(\Gamma \setminus P) = X \setminus \{P\}$ and $\operatorname{edge}(\Gamma \setminus P) = Y \setminus (Y_p \cup \overline{Y_p})$.

Proposition 2.5. *Let P be a non-isolated terminal vertex of a graph Γ.*

a) Γ is connected if and only if $\Gamma \setminus P$ is connected,
b) Every circuit of Γ is contained in $\Gamma \setminus P$,
c) Γ is a tree if and only if $\Gamma \setminus P$ is a tree.

Proposition 2.6. *Let Γ be a tree of finite diamter n.*

a) The set of terminal verticies of Γ is non-empty.
b) If $n \geq 2$ then $\operatorname{vert}(\Gamma) \setminus t(\Gamma)$ generates a subtree Γ' of diameter $n-2$.
c) If $n = 0$ then $\Gamma \cong \mathrm{Path}_0$ and if $n = 1$ then $\Gamma \cong \mathrm{Path}_1$.

Proof. Since item c is obvious and b and c combine to prove a we only prove b. To this end let $X' = \operatorname{vert}\Gamma \setminus t(\Gamma)$. If $P, Q \in X'$ then any point on the geodesic from P to Q is non-terminal and so $\operatorname{vert}\Gamma' = X'$. Moreover if $\ell(P,Q) = m$, we can extend the geodesic to one of length $m+2$ in X by appending a vertex to each end and hence $m \leq n-2$. Finally the diameter of Γ' is at least $n-2$ since one can take a geodesic of legnth n in X and upon removing the extremities get a geodesic of length $n-2$ which must be in Γ'. □

We remark here that any automorphism of Γ restricts to an automorphism of Γ' with notation as above. Thus by induction one gets the following corollary.

Corollary 2.7. *A tree of even diameter (resp. odd diameter) has a vertex (resp. edge) which is invariant under all automorphisms.*

To conclude this chapter, we discuss subtrees of graphs. For a non-empty graph Γ, subgraphs of Γ which are trees can be partially-ordered with respect to inclusion and by Zorn's Lemma there exists a maximal element. Such an element is a *maximal tree* of Γ.

Proposition 2.8. *Let Λ be a maximal tree of a connected graph Γ. Then* $\operatorname{vert}\Lambda = \operatorname{vert}\Gamma$.

Proof. If not, since Γ is connected there exists an edge of Γ with origin in Λ but terminus outside of Λ. But then the graph Λ together with this edge is still a tree contradicting maximality of Λ. □

Proposition 2.9. *Let Γ be a connected graph with a finite vertex set. Setting,*

$$s = |\operatorname{vert}\Gamma|, \quad a = \frac{1}{2}|\operatorname{edge}\Gamma|,$$

one has $a \geq s - 1$ with equality if and only if Γ is a tree.

Proof. If Γ is a finite tree then Γ can be obtained by repeated adjoining of a geometric edge to a terminal vertex, starting from the tree with a single vertex. Consequently since the equality holds for the tree consisting of a single vertex, and remains true after this addition of a single geometric edge, it holds for all finite trees.

Generally, if Γ is non-empty and Γ' is a maximal tree, the previous gives $s(\Gamma) = s(\Gamma')$ and of course $a(\Gamma) \geq a(\Gamma')$ with equality if and only if $\Gamma = \Gamma'$. Since Γ' is a tree, $a(\Gamma') = s(\Gamma') - 1$ and hence,

$$a(\Gamma) = s(\Gamma) - 1 + (a(\Gamma) - a(\Gamma')) \geq s(\Gamma) - 1.$$

□

Let Γ be a connected, non-empty graph, $\Lambda \subseteq \Gamma$ a subgraph which is a disjoint union of tree Λ_i. We aim to construct a graph Γ/Λ whose realization is the quotient of $\operatorname{real}\Gamma$ obtained by identifying each Λ_i to a point. To this end define $\operatorname{vert}\Gamma/\Lambda$ to be the quotient of $\operatorname{vert}\Gamma$ which identifies vertices of each Λ_i toa single point. Define then the edge set $\operatorname{edge}\Gamma/\Lambda := \operatorname{edge}\Gamma \setminus \operatorname{edge}\Lambda$. Finally the map which assigns endpoints to edges is the obvious one induced by passing endpoints through the quotient.

Proposition 2.10. *The projection* $\operatorname{real}\Gamma \to \operatorname{real}\Gamma/\Lambda$ *is a homotopy equivalence.*

Proof. Fixing notation, let $X = \operatorname{real}\Gamma, A = \operatorname{real}\Lambda, A_i = \operatorname{real}\Lambda_i$. Each A_i is contractible and so there is a homotopy $h_t : A \to A$ such that $h_0 = \operatorname{Id}_A$ and h_1 contracts each A_i to a point. Now, A is a subcomplex of X and so by Proposition 2.4 a lift exists $H_t : X \to X$ such that $H_0 = \operatorname{Id}_X$ and $H_t|_A = h_t$. At time $t = 1$, we have the map H_1 which factors through the quotient map $p : X \to Y := \operatorname{real}\Gamma/\Lambda$. More explicitly we have a map $f : Y \to X$ such that $H_1 = f \circ p$ and therefore $f \circ p \sim H_0 = \operatorname{Id}_X$.

It remains to show $p \circ f \sim \operatorname{Id}_Y$. Since for every $t \in [0, 1]$, we have $H_t(A_i) \subseteq A_i$, we can define a map H'_t such that

$$\begin{array}{ccc} X & \xrightarrow{H_t} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[H'_t]{} & Y \end{array}$$

commutes. Then for $t = 1$ we get the diagram,

$$\begin{array}{ccc} X & \xrightarrow{H_1} & X \\ {\scriptstyle p}\downarrow & \overset{f}{\nearrow} & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[H'_1]{} & Y \end{array}$$

which commutes since the upper triangle and the outer square commute. Therefore $p \circ f = H'_1 \sim H'_0 = \mathrm{Id}_Y$ completing the proof. □

We remark now that since maximal trees contain all vertices of a graph, for a maximal tree Λ of Γ, Γ has the homotopy type of a bouquet of circles by the proposition.

3. Trees and Free Groups

Let X be a graph on which the group G acts. We say that G acts *without inversion* if for every $y \in \operatorname{edge}\Gamma$ we have that $gy \neq \overline{y}$ for all $g \in G$. Equivalently this is to say that the action of G preserves an orientation of X. Provided G acts without inversion, we obtain the quotient graph $G\backslash X$ via the obvious construction. Moreover the realization of $G\backslash X$ is the quotient of the realization of X by the induced action.

Proposition 3.1. *Let X be a connected graph and G a group acting on X without inversion. Then every subtree T' of $G\backslash X$ lifts to a subtree of X.*

Proof. Let Ω denote the collection of subtrees of X which project injectively into T'. By Zorn's Lemma, Ω has a maximal element, say T_0 and let T'_0 denote its image in $G\backslash X$. Suppose by way of contradiction that $T'_0 \neq T'$. Then there exists an edge $y' \in T'$ such that $o(y') \in T'_0$ and $P := t(y') \notin T'_0$. L Let y be an edge lifting y', which we may assume to be such that $o(y) \in T_0$. We then let T_1 denote the graph obtained from adjoining y and $t(y)$ to T_0, which is then an element of Ω strictly containing T_0, contradicting maximality. Thus we conclude $T'_0 = T'$ and hence T_0 is our desired lift. □

We define a *tree of representatives* of X mod G to be a lift of a maximal tree in $G\backslash X$. We recall an easy but nevertheless important theorem.

Theorem 3.2. *Let G be a group, $S \subseteq G$ a subset and $\Gamma = \Gamma(G, S)$ the corresponding graph. Then Γ is a tree if and only if G is free and S is a free basis for G.*

Throughout we will assume our groups G act without inversion unless we explicitly state otherwise. Thus when we say G acts freely on a graph X we mean that G acts freely on the vertex set and it acts without inversion. We now show that free groups are the only groups which can act freely on trees. More precisely we have the following theorem.

Theorem 3.3. *Let G be a group acting freely on a tree X. Let Y_+ be an orientation preserved by G and let T be a tree of representatives for X mod G.*

- *a) Let S denote the set of $g \neq 1$ in G such that there exists an edge $y \in Y_+$ such that $o(y) \in T$ and $t(y) \in gT$. Then S is a free basis for G.*
- *b) If X^* has finitely many vertices s and if $\operatorname{Card}(\operatorname{edge} X^*) = 2a$ then we have $\operatorname{Card}(S) - 1 = a - s$.*

Proof. Since G acts freely and T is a tree of representatives we have that $g \mapsto gT$ is a bijection and for any $g \neq g'$, $gT \cap g'T = \varnothing$. Consider then the tree $X' = X/GT$ which is the tree obtained by contracting each translate gT to a single point. Proposition 2.10 gives that X' is itself a tree. Thus the inverse of the bijection $g \mapsto gT$ should be thought of as a bijection between vertex sets

$$\alpha : \operatorname{vert} X' \simeq \operatorname{vert} \Gamma(G,S) = G.$$

If we can extend α to a isomorphism of graphs, we will obtain that $\Gamma(G,S)$ is a tree and so it is free with S its free basis.

Giving X' the orientation Y'_+ induced by the orientation Y_+ (since $\operatorname{edge} X' = \operatorname{edge} X \setminus \operatorname{edge}(GT)$), it then suffices to define an a map $\alpha : Y'_+ \to G \times S = (\operatorname{edge} \Gamma(G,S))_+$. To this end let $y \in Y'_+$ and let $gT = o(y)$, $g'T = t(y)$. Since in X the edge y connects a point from gT to a point in $g'T$ we conclude that $s = g^{-1}g' \in S$. We therefore define

$$\alpha(y) = (g,s).$$

α is surjective by the definition of S and injective since the action is free. Thus α is the desired isomorphism of graphs, completing the proof of *a*.

Part *b* is simply a calculation of Euler characteristics. We know that $\chi(G\backslash X) = s - a$ and so it suffices to check that that $\chi((G\backslash X)/T*) = 1 - \operatorname{Card}(S)$ since both spaces are homotopy equivalent and thus have identical Euler characteristic. Of course we know that $(G\backslash X)/T*$ has a single vertex so we simply need to show that it has $\operatorname{Card}(S)$ as the cardinality of an oriented edge set. To this end note that $\operatorname{Card}(S) = |\{y \in Y_+ | o(y) \in T \text{ and } t(y) \notin T\}|$. Now to see that an edge $y \in Y_+$ such that $o(y) \in T$ and $t(y) \notin T$ corresponds in bijection to the edges in $(G\backslash X)/T*$ note that any edge in T corresponds bijectively to an edge in T^* which is contracted to a point in $(G\backslash X)/T*$. The same holds true for any edge in gT for any $g \in G$ since T and gT both project bijectively onto T^* in the quotient. Hence we only need to count the edges which move across translates in X but since an edge y connecting gT to $g'T$ yields and edge in the same orbit as y, namely $g^{-1}y$ from T to $g^{-1}g'T$ we see that we need only count the edges which start in T. This justifies *b*. $\square$

The theorem does more than just say that only free groups act freely on trees, as it provides a generating set. Nevertheless we provide here a simpler topological proof that only free groups act freely on trees. Since G acts freely on X, it acts freely and properly discontinuously on the realization $\operatorname{real}(X)$. Hence the map $\operatorname{real}(X) \to \operatorname{real}(G\backslash X)$ is a covering map with and $\pi_1(G\backslash X) = G$. Moreover, we know that upon contracting a maximal tree T^* of X to a point we obtain a bouquet of circles and since

$$G = \pi_1(G\backslash X) = \pi_1((G\backslash X)/T^*)),$$

we conclude that G is free.

Of course if one does not want to compute an explicit basis for G then there is a much easier topological proof that only free groups can act freely on trees as follows: A free action of G on a tree X yields a free action on the realization $\overline{X} = \operatorname{real} X$ and since we assume to free to imply the action is without inversion, we have that this is a free and properly discontinuous action of G on the simply connected $\overline{X}$ which means $\overline{X} \to G\backslash\overline{X}$ is a covering map and $\pi_1(G\backslash\overline{X}) = G$. But of course, for a maximal tree $T \subseteq G\backslash\overline{X}$ we have a homotopy equivalence between $G\backslash\overline{X}$ and its quotient mod T, with the latter having homotopy type a bouquet of circles. Thus G is a free group.

To wrap up this section we discuss consequences of Theorem 3.3.

Corollary 3.4. *Every subgroup H of a free group G is free.*

Proof. G acts freely on its Cayley graph which is a tree since G is free. Then the restricted action of H on the Cayley graph is also free, whence H is free. □

Next we have the Schreier Index Formula, which while it really has a combinatorial proof, should more simply be thought of due to the fact that the Euler characteristic of a finite n-sheeted covering is n times the Euler characteristic of the base space.

Proposition 3.5. *Let G be a free group, $H \leq G$ of index $n < \infty$. Then if r_H, r_G denote the rank H and G respectively, one has*

$$r_H - 1 = n(r_G - 1).$$

A more explicit result is the following.

Proposition 3.6. *Let G be a free group with basis S and $H \leq G$.*

a) One can choose a set T of representatives of the cosets of $H\backslash G$ such that if $t \in T$ has the unique decomoposition

$$t = s_1^{\epsilon_1} \dots s_n^{\epsilon_n}, \quad \epsilon_i = \pm 1 \quad \textit{and} \quad \epsilon_i = \epsilon_{i+1} \quad \textit{if} \quad s_i = s_{i+1},$$

then each partial product $s_1^{\epsilon_1}, s_1^{\epsilon_1}s_2^{\epsilon_2}, \dots, s_1^{\epsilon_1} \dots s_{n-1}^{\epsilon_{n-1}}$ also belongs to T.

b) With T as above, set $W = \{(t,s) \in T \times S : ts \notin T\}$. Then if $(t,s) \in W$ the set $h_{t,s} = tsu^{-1}$ where $u \in T$ is such that $Hts = Hu$ is a free basis for H.

Proof. a) Let Γ be the Cayley graph associated to G and S given the orientation $G \times S$. H acts on Γ freely and a choice of representatives T is equivalent to a choice of a tree of representatives of Γ mod H which contains 1. Such lifts exists giving the result.

b) For Λ a maximal tree in $H\backslash\Gamma$, with vert $\Lambda = T$ as above, Theorem 3.3 gives that a generating set of H consists of $h \neq 1$ in H such that there exists an edge $(t,s) \in G \times S$ with $t \in T$ and $ts \in hT$. Then for $u = h^{-1}ts$ we gave $Hts = Hu$ and $h_{t,s} = h$ giving a correspondence between W and the generating set from Theorem 3.3. □

Part *b)* of the above is nevertheless not the most efficient manner to calculate the generating set of a subgroup of a free group. Instead one typically follows the folding procedure outlined in Stalling's *The Topology of Finite Graphs.*

4. Trees and Amalgams

4.1. **Amalgams of two Groups.** For the remainder of these notes we will assume that our groups act without inversion except in the case where it is explicitly stated otherwise. For a group G acting on a graph X, a *fundamental domain* of X mod G is a subgraph T of X isomorphic to $G\backslash X$ under the projection map. Note that if G acts on a tree, a fundamental domain will exist if and only if $G\backslash X$ is a tree. In this first section we consider actions whose fundamental domain is a *segment*, i.e. a graph isomorphic to a path of length 1. The following two theorems classify such actions on trees.

Theorem 4.1. *Let G be a group acting on a graph X with fundamental domain a segment T, having endpoints P and Q with y the directed edge from P to Q. If G_P, G_Q, G_y are the stabilizer of P, Q, and y respectively, then X is a tree if and only if the natural homomorphism $G_P *_{G_y} G_Q \to G$ induced via inclusions is an isomorphism.*

On the other hand, every amalgam has a unique action on a tree yielding a segment as its fundamental domain.

Theorem 4.2. *Suppose $G = G_1 *_A G_2$. Then there exists a tree X on which G acts with fundamental domain a segment T with endpoints P, Q and directed edge y such that $G_P = G_1, G_Q = G_2$, and $G_y = A$. Moreover X is unique up to isomorphism.*

Theorem 4.1 is a consequence of the following two lemma.

Lemma 4.3. *X is connected if and only if G is generated by $G_P \cup G_Q$.*

Proof. Let X' be the connected component of X containing T, G' be the set of $g \in G$ such that $gX' = X'$, and G'' be the subgroup generated by $G_P \cup G_q$. We aim to show $X = X'$ if and only if $G = G''$. If $h \in G_P \cup G_Q$, then T and hT share a common vertex, and so $hT \subseteq X'$ and thus $hX' = X'$. Thus $h \in G'$ and so $G'' \subseteq G'$. On the other hand note that $G''T$ and $(G \setminus G'')T$ are disjoint subgraphs whose union is X. Hence $X' \subseteq G''T$ and so $G'' \subseteq G'$. Thus we conclude $G'' = G'$ and since $X = X'$ if and only if $G = G' = G''$ we obtain the result. □

Lemma 4.4. *X contains no circuit if and only if the map $G_P *_{G_y} G_Q \to G$ is injective.*

Proof. X contains a circuit if and only if there is a path $c = (w_0, \ldots, w_n)$ without backtracking such that $o(w_0) = t(w_n)$. Since T is a fundamental domain, each $w_i = h_i y_i$ for some $h_i \in G$ and $y_i = y$ or $\overline{y}$. Upon passing to the quotient we further note that $\overline{y_i} = y_{i-1}$ for all $1 \leq i \leq n$. We set $P_i = o(y_i) = t(y_{i-1})$. *So in particular $P_i = P$ if $y_i = y$ and is Q otherwise.* Then $h_i = h_{i-1} g_i$ for some $g_i \in G_{P_i}$ since

$$hP_i = h_i o(y_i) = o(h_i y_i) = t(h_{i-1} y_{i-1}) = h_{i-1} t(y_{i-1}) = h_{i-1} P_i.$$

Moreover $g_i \notin G_y$ since

$$\overline{h_i y_i} \neq h_{i-1} y_{i-1}.$$

Now since $o(w_0) = t(w_n)$ if and only if $t(y_n) = P_0$ which occurs if and only if

$$h_0 P_0 = h_n P_0 = h_0 g_1 \ldots g_n P_0, \quad \text{i.e. } g_1 \ldots g_n \in G_{P_0},$$

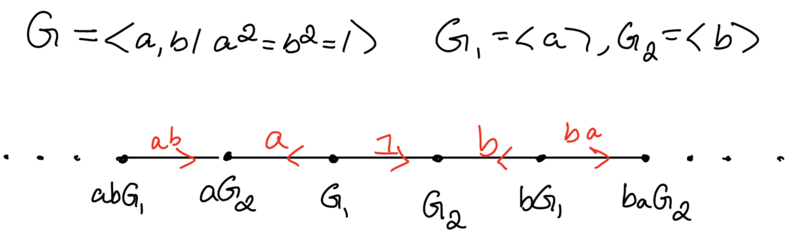
We then conclude that X contains a circuit if and only if there exists a sequence of verticies $P_0, \ldots, P_n \in T$ with $\{P_i, P_{i+1}\} = \{P, Q\}$ for all i and a sequence either a sequence $g_i \in G_{P_i} \setminus G_y$ with $g_0 g_1 \ldots g_n = 1$, or if $g_0 \in G_{P_0}$ then we get a sequence such that $(g_0 g_1) g_2 \ldots g_n = 1$. Either case is a reduced word in the amalgam which is trivial in G from which the result follows. □

Proof of Theorem 4.2. It is clear that the graph X must be such that

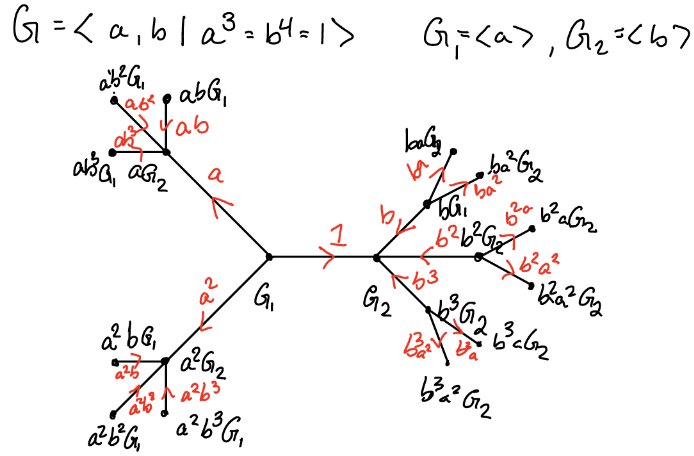
$$\operatorname{vert} X = G/G_1 \sqcup G/G_2, \quad \operatorname{edge} X = G/A \sqcup \overline{G/A},$$

where the maps $o : G/A \to G/G_1, t : G/A \to G/G_2$ are the inclusion maps. This graph is then a tree by Theorem 4.1 □

Let us now illustrate the trees in Theorem 4.2 via two examples. First consider the infinite dihedral group $D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.



Similarly we present the tree for $G = \mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/4\mathbb{Z}$.



For the remainder of this section let $G = G_1 *_A G_2$ and X the corresponding tree as in Theorem 4.2.

Proposition 4.5. *Suppose Γ is a subgroup of $G = G_1 *_A G_2$ such that $\Gamma \setminus \{1\}$ does not meet any conjugate of G_1 or G_2. Then Γ is a free group.*

Proof. Since conjugates of G_1 and G_2 are precisely the stabilizers of the vertices of X, it then follows that Γ acts freely on the tree X and is therefore free. $\square$

We say a subset of G is *bounded* if there is a bound on the length of any elements reduced decomposition.

Theorem 4.6. *Any bounded subgroup of G is contained in either a conjugate of G_1 or a conjugate of G_2.*

Proof. Say $g = g_1 h_1 \ldots g_n h_n$ with $g_i \in G_1, h_i \in G_2$ and let P, Q denote the vertices corresponding to G_1, G_2 in X respectively. Since $h_n \in G_2$, we have that $d(h_n P, Q) = 1$. Now by acting on the path from P to Q then Q to $h_n P$ by the element g_n, we see that $d(g_n h_n Q, P) = 2$. Continuing inductively we find that $d(g_1 h_1 \ldots g_n h_n Q, P) = 2n$. Hence for any bounded subset $\Sigma \subseteq G$, we have $\Sigma(\operatorname{vert} T)$ is bounded where T is the fundamental domain of X under its G action. It thus suffices to prove the following proposition.

Proposition 4.7. *Let Γ be a group acting on a tree X. The following are equivalent:*

a) For every bounded subset A of $\operatorname{vert} X$, ΓA is bounded.
b) There is a $P \in \operatorname{vert} X$ such that ΓP is bounded.
c) There is a vertex of X invariant under Γ.

Proof of Proposition 4.7. The implication *a)* $\Rightarrow$ *b)* is immediate. For *b)* $\Rightarrow$ *c)* let Y be the subtree generated by ΓP where ΓP is bounded. This is a bounded subtree which is stable under the action of Γ. Thus there is either an edge of a vertex which is stabilized and since Γ acts without inversion, if there is an edge stabilizes, then Γ stabilizes its extremities.

For *c)* $\Rightarrow$ *a)* assume P is left invariant under Γ and note that upon realizing Γ as an inverse system with P as the singled out point then one must have points of distance n from P are mapped to points of distance n from P under the action. Then for any bounded subset A of X say with diameter bounded above by n, if m denotes the minimum distance from P to a point in X, we have that no point in A is at distance more than $n+m$ from P. Thus the same is true for ΓA and so ΓA has diameter at most $2n+2m$. □

The implication *b)* $\Rightarrow$ *c)* together with the observation before the proposition gives the result of Theorem 4.6. □

4.2. **Trees of Groups.** A *graph of groups* (G,T) consists of a graph T together with a group G_P for each $P \in \operatorname{vert} Y$ and a group G_y for each $y \in \operatorname{edge} T$ along with monomorphisms $G_y \to G_{t(y)}$ for each y. Moreover we insist that $G_y = G_{\overline{y}}$. If T is a tree, then (G,T) is said to be a *tree of groups.* We then let

$$G_T = \lim_{\rightarrow}(G,T),$$

denote the corresponding direct limit.

For some examples, note if T is simply a segment with vertices P,Q and an edge y then $G_T = G_P *_{G_y} G_Q$. More generally, if the tree T is obtained from the tree T' by adjoining a terminal vertex P along an edge $\{y,\overline{y}\}$ then

$$G_T = G_{T'} *_{G_y} G_P \qquad \text{where } G_{T'} = \lim_{\rightarrow}(G,T').$$

Finally if $(G_i)_{i\in I}$ is a family of groups and A is group coming with monomorphisms $A \to G_i$ for each i we form the corresponding tree T: its vertex set consists of elements of I whose vertices are labelled by G_i and a single element $0 \notin I$ which is labelled by A. Then the edge set consists of the pairs $(i,0)$ and $(0,i)$ for all i, with edges labelled by A. We then have the tree of groups with this assignment of groups to vertices and edges, together with the monomorphisms $A \to A, A \to G_i$. Then

$$G_T = *_A G_i.$$

We now have the tools to classify the structure of group actions on trees whose fundamental domain is again a tree. (The next section will handle the completely general case).

Theorem 4.8. *Let (G,T) be a tree of groups. There is a graph X containing T and an action of G_T on X which is characterized up to isomorphism by the following: T is a fundametal domain for X mod G_T and for all $P \in \operatorname{vert} T, y \in \operatorname{edge} T$, $\operatorname{Stab} P = G_P, \operatorname{Stab} y = G_y$. Moreover X is a tree.*

Proof. Necessarily

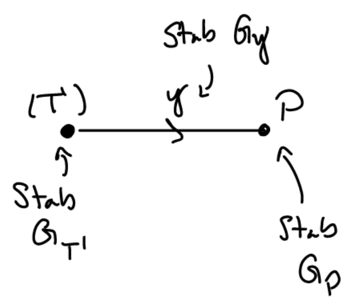
$$\operatorname{vert} X = \bigsqcup_{P \in \operatorname{vert} T} G_T/G_P, \quad \operatorname{edge} X = \bigsqcup_{y \in \operatorname{edge} T} G_T/G_y,$$

with extremity maps given by the inclusions $o : G_y \to G_{o(y)}, t : G_y \to G_{t(y)}$. All assertions of the theorem are then clear except for the conclusion that X is a tree. For this we claim it suffices to consider the case where T is finite since then we will appeal to the fact that T is the direct limit of its finite subtrees and similarly G_T is the direct limit of $G_{T'}$ where $T' \subseteq T$ finite and similarly for the corresponding graphs X'.

Now supposing T is finite we will argue by induction on $|\operatorname{vert} T|$ with the base case being trivial. For $y \in \operatorname{vert} T$ with $P = t(y)$ a terminal vertex, we obviously obtain T by adjoining P along y to the tree $T' = T \setminus P$. For $G_{T'} = \lim_{\to}(G, T')$ we know that

$$G_T = G_{T'} *_{G_y} G_P.$$

Setting $X' = G_T T'$ we see immediately that X' is the graph corresponding the to tree of groups (G, T') and by induction X' is a tree. We then claim that for any $g \neq h \in G_T/G_{T'}$, gX', hX' are pairwise disjoint. Contracting these subtrees to a point one obtains the graph $\widetilde{X}$ on which G_T acts yielding fundamental domain a segment:



Since we already know $G_{T'} *_{G_y} G_P \to G_T$ is an isomorphism we obtain by Theorem 4.1 that $\widetilde{X}$ is a tree. Moreover since $\widetilde{X}$ is homotopy equivalent to X, this implies X is also a tree. □

Conversely we can start from a group acting on a graph X with fundamental domain a tree T. We get a tree of groups (G, T) by associating to each vertex in T its point stabilizer under the action and similarly for the edges. The morphisms are then the obvious ones induced by the inclusions $G_y \to G_{t(y)}$. Let $G_T = \lim_{\to}(G, T)$. The inclusion maps $G_P \to G$ then extend to a homomorphism $G_T \to G$ and this homomorphism is surjective if X is connected. Meanwhile if $\widetilde{X}$ denotes the tree associated with (G, T) as in Theorem 4.8, the identity map $T \to T$ extends to a morphism $\widetilde{X} \to X$ which is equivariant to the G_T and G actions.

Theorem 4.9. *With the notation and hypotheses as above, the following are equivalent:*

a) X *is a tree.*
b) $\widetilde{X} \to X$ *is an isomorphism.*
c) $G_T \to G$ *is an isomorphism.*

Proof. Note that *c)* $\Rightarrow$ *b)* and *b)* $\Rightarrow$ *a)* are given by Theorem 4.8. Another way to think of *c)* $\Rightarrow$ *b)* is that $\widetilde{X}$ has edges and vertices labeled by cosets of G whereas X has edges and vertices labelled by orbits of G, so since the groups are the same, the result follows from the orbit stabilizer theorem.

For *b)* $\Rightarrow$ *c)*, let $P \in \operatorname{vert} T$ and let $(G_T)_P$ (resp. G_P) denote the corersponding stabilizers in G_T and G_P. By construction the homomorphism $G_T \to G$ induces isomorphisms from each $(G_T)_P$ to G_P. On the other hand letting f denote the map $G_T \to G$, we claim there is some P such that $\operatorname{Ker} f \subseteq (G_T)_P$. If not, then for some $g \in \operatorname{Ker} f \setminus (G_T)_P$ we'd have $(G_T)_P$ and $g(G_T)_P$ correspond to different vertices in $\widetilde{X}$ both of which map to P under the map $\widetilde{X} \to X$, contradicting bijectivity of $\widetilde{X} \to X$. Since we already f is surjective, we conclude that it is an isomorphism.

For *a)* $\Rightarrow$ *c)* note since $G_T T = \widetilde{X}$ and $GT = X$ we have $\widetilde{X} \to X$ is surjective. For injectivity note certainly the map $\widetilde{X} \to X$ is locally injective (i.e. injective on the set of edges originating from a common vertex.) Thus we only need the following lemma.

Lemma 4.10. *Let* $f : \widetilde{X} \to X$ *be a locally injective morphism from a connected graph* $\widetilde{X}$ *into a tree* X*. Then* f *is injective.*

Proof of Lemma 4.10. Suppose $P, Q \in \operatorname{vert} \widetilde{X}$ are such that $f(P) = f(Q)$. Being connected we have a path without backtracking $(y_1, \ldots, y_n)$ from P to Q in $\widetilde{X}$. If $n = 0$ then $P = Q$ and if $n = 1$ then $f(y_1)$ is a loop in X, contradicting X being a tree. If $n > 1$, then $(f(y_1), \ldots, f(y_n))$ is a circuit in X and since X is a tree this cannot be without backtracking. If i is such that $\overline{f(y_i)} = f(y_{i+1})$ we then notice that since $\overline{f(y_i)} = f(\overline{y_i})$, local injectivity forces $\overline{y_i} = y_{i+1}$ since both have the same origin. This contradicts our assumption that the original path was without backtracking. Hence f is injective on the set of vertices and this together with local injectivity implies that f is also injective on the set of edges. $\square$

$\square$

To conclude this section, let us provide a good example to keep in mind as to why this theorem fails in the case where X is not a tree. We have a natural action of D_3 on the complete graph on 3 vertices and consequently on the barycentric subdivision of this graph. Let X be this barycentric subdivision of the complete graph on 3 vertices. Under this action the fundamental domain is simply segment with vertex stabilizers $\mathbb{Z}/2\mathbb{Z}$ and edge stabilizer $\{1\}$. But G_T here is then $D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \ncong D_3$ and the graph corresponding to D_∞ is certainly not the graph X, so Theorem 4.9 fails in this case.

5. Structure of a Group Acting on a Tree

5.1. **Fundamental Group of a Graph of Groups.** Having already classified the structure of groups acting on trees which also have fundamental domain a tree, we now build up the theory in order to classify the completely general case of an arbitrary group acting on an arbitrary tree. First we discuss the fundamental group of a graph of groups. Let Y be a a connected non-empty graph, (G, Y) a graph of groups. Recall for each edge $y \in \operatorname{edge} Y$ this carries the information of two monomorphisms,

$$G_y \to G_{o(y)}, \quad G_y \to G_{t(y)}.$$

For $a \in G_y$ we recall our notation that a^y denotes the image of a in $G_{t(y)}$ under the above monomorphisms. Let $F\{\operatorname{edge} Y\}$ be a free group with free basis $\operatorname{edge} Y$ and set,

$$\Gamma = (*_{P \in \operatorname{vert} Y}\ G_P) * F\{\operatorname{edge} Y\}.$$

Let $F(G, Y) = \Gamma/N$ where N is the normal subgroup generated by elements of the form,

$$y\overline{y}, \quad ya^y y^{-1}(a^{\overline{y}})^{-1},$$

where $y \in \operatorname{edge} Y, a \in G_y$. In other words, $F(G, Y)$ is generated by G_P and $\operatorname{edge} Y$ subject to the relations in each G_P and the added relations $\overline{y} = y^{-1}$ and $ya^y y^{-1} = a^{\overline{y}}$ for every $y \in \operatorname{edge} Y, a \in G_y$. Now let c be a path in Y whose origin is P_0 and let $(y_1, \ldots, y_n)$ denote the edges of c. Set

$$P_i = o(y_{i+1}) = t(y_i).$$

A *word* of type c in $F(G, Y)$ is then a pair (c, μ) where $\mu = (r_0, \ldots, r_n)$ is a sequence of elements with each $r_i \in G_{P_i}$. The element,

$$|c, \mu| = r_0 y_1 r_1 y_2 \ldots y_n r_n \in F(G, Y),$$

is then associated to the word (c, μ).

With these definitions and notations fixed, we now provide two definitions for the *fundamental group of* (G, Y) and prove the equivalence (up to isomorphism) of their definitions. For our first definition fix a basepoint $P_0 \in \operatorname{vert} Y$. Let $\pi_1(G, Y, P_0)$ be the set of elements of $F(G, Y)$ of the form $|c, \mu|$ where c is a circuit which starts and ends at P_0. $\pi_1(G, Y, P_0)$ is then easily seen to be subgroup of $F(G, Y)$ and is the fundamental group of (G, Y) based at P_0. Note that when I is the trivial graph of groups, i.e. we assign the trivial group to every vertex, we simply obtain the ordinary fundamental group $\pi_1(Y, P_0)$ and the canonical morphism $G \to I$ extends to a surjective homomorphism,

$$\pi_1(G, Y, P_0) \to \pi_1(Y, P_0),$$

whose kernel is the normal subgroup generated by each of the G_P.

For our second definition, let T be a maximal tree of Y. The fundamental group $\pi_1(G, Y, T)$ with respect to T is then the quotient of $F(G, Y)$ by the normal subgroup generated by $y \in \operatorname{edge} T$. I.e. $\pi_1(G, Y, T)$ is generated by the G_P and $y \in \operatorname{edge} Y$ subject to the relations of each G_P along with,

$$y\overline{y}, \quad ya^y y^{-1}(a^{\overline{y}})^{-1},$$

and

$$y = 1 \quad \text{ if } y \in \operatorname{edge} T.$$

To avoid confusion, from now on let us write g_y for the element corresponding to y in $F(G, Y)$ or any of its quotients.

Before discussing the equivalence of these definitions, let us provide some examples. First if $G_y = \{1\}$ for every edge in Y and A denotes an orientation of Y, then $\pi_1(G,Y,T)$ is generated by the G_P and g_y for $y \in A \setminus (T \cap A)$ subject to no relations other than those in the G_P. That is,

$$\pi_1(G,Y,T) = (*_{P \in \operatorname{vert} Y} G_P) * F\{A \setminus (T \cap A)\}.$$

If Y is a segment with vertices, P, Q and connecting edge y then it is clear that $\pi_1(G,Y,Y) = G_P *_{G_y} G_Q$. More generally if Y is a tree, then

$$\pi_1(G,Y,Y) = \varinjlim (G,Y).$$

For our last example suppose Y is a loop with lone vertex P and edge denoted y. Setting $A = G_y$ we then have two monomorphisms $A \xrightarrow{y} G_P, A \xrightarrow{\overline{y}} G_P$. Then $F(G,Y) = \pi_1(G,Y,P)$ is generated by G_P (and its relations) along with an element $g = g_y$ such that

$$g a^y g^{-1} = a^{\overline{y}},$$

for any $a \in G_y$. Identifying A with the subgroup of G_P defined via the assignment $a \mapsto a^y$ and letting θ denote the map $a \mapsto a^{\overline{y}}$ we then see that

$$\pi_1(G,Y,P) = \langle \mathscr{S}_P, g | \mathscr{R}_P, g a^y g^{-1} = \theta(a) \rangle,$$

which we recognize as the HNN extension of G_P with respect to the two monomorphisms $A \to G_P$.

To conclude this section we establish the equivalence between our two definitions of the fundamental group of a graph of groups.

Proposition 5.1. *Let (G,Y) be a graph of groups, $P_0 \in \operatorname{vert} Y$ and T a maximal tree. The canonical projection map $F(G,Y) \to \pi_1(G,Y,T)$ induces an isomorphism,*

$$p : \pi_1(G,Y,P_0) \to \pi_1(G,Y,T).$$

Proof. If $P \in \operatorname{vert} Y$ let c_P denote the unique geodesic in T connecting P_0 to P. Let $y_1, \ldots, y_n$ denote the edges of c_P and set

$$\gamma_P = y_1 \ldots y_n \in F(G,Y).$$

Likewise we set

$$\begin{aligned} x' &= \gamma_P x \gamma_P^{-1}, \\ y' &= \gamma_{o(t)} y \gamma_{t(y)}^{-1}. \end{aligned}$$

If $y \in \operatorname{edge} T$ then either $c_{t(y)} = (c_{o(y)}, y)$ or $c_{o(y)} = (c_{t(y)}, \overline{y})$ since otherwise $(c_{o(y)}, y, c_{t(y)})$ would be a loop in T without backtracking. In either case $y' = 1$. On the other hand $(\overline{y}')y' = 1$ for any $y \in \operatorname{edge} Y$ and if $a \in G_y$ then,

$$\begin{aligned} y'(a^y)'{y'}^{-1} &= \gamma_{o(y)} y \gamma_{t(y)}^{-1} \gamma_{t(y)} a^y \gamma_{t(y)}^{-1} \gamma_{t(y)} y^{-1} \gamma_{o(y)}^{-1}, \\ &= \gamma_{o(y)} y a^y y^{-1} \gamma_{o(y)}^{-1}, \\ &= \gamma_{o(y)} a^{\overline{y}} \gamma_{o(y)}^{-1}, \\ &= (a^{\overline{y}})'. \end{aligned}$$

Clearly elements of the form x', y' belong to $\pi_1(G,Y,P_0)$ and the above relations verify that we have a (unique) homomorphism $f : \pi_1(G,Y,T) \to \pi_1(G,Y,P_0)$ such that $x \mapsto x'$ for $x \in G_P$ and $y \mapsto y'$ for $y \in \operatorname{edge} Y$.

Now $p(\gamma_P) = 1$ from which it follows that $p \circ f = \mathrm{Id}$. To see that $f \circ p = \mathrm{Id}$, let c be a closed path with origin P_0 and edges $y_1, \ldots, y_n$. Similarly label the vertices,

$$P_i = o(y_{i+1}) = t(y_i).$$

Let (c, μ) with $\mu = (r_0, \ldots, r_n)$ be a word of type c, having corresponding element

$$|c, \mu| = r_0 y_1 r_1 y_2 \ldots r_n y_n.$$

Then,

$$r'_i = \gamma_{P_i} r_i \gamma_{P_i}^{-1}, \quad y'_i = \gamma_{P_i} y_i \gamma_{P_{i+1}}^{-1}.$$

So,

$$r'_0 y'_1 r'_1 y'_2 \ldots r'_n y'_n = \gamma_{P_0} (r_0 y_1 r_1 y_2 \ldots r_n y_n) \gamma_{P_0}^{-1} = r_0 y_1 r_1 y_2 \ldots r_n y_n.$$

Hence $f \circ p = \mathrm{Id}$, completing the proof. □

Having defined the fundamental group of a graph of groups, we can build up a connection between $\pi_1(G, Y, T)$ and the ordinary topological fundamental group $\pi_1(Y, T)$. We start with an arbitrary non-empty connected graph of groups (G, Y) and T a maximal tree of Y. We let $\widetilde{Y}$ denote the universal cover of Y and let $(\widetilde{Y}, T)$ denote $\widetilde{Y}$ with a chosen lift of T to $\widetilde{Y}$. Then $\pi_1(Y, T)$ acts freely by deck transformations on the tree $\widetilde{Y}$. We define a graph of groups $(G, \widetilde{Y})$ via if $Q \in \operatorname{vert} \widetilde{Y}$ projects to $P \in \operatorname{vert} Y$ we set $G_Q = G_P$ and assign the edge groups similarly. Our claim is that $\pi_1(G, Y, T)$ is isomorphic to a semidirect product,

$$\pi_1(G, Y, T) \cong \pi_1(G, \widetilde{Y}, \widetilde{Y}) \rtimes \pi_1(Y, T) = \lim_{\rightarrow} (G, \widetilde{Y}) \rtimes \pi_1(Y, T),$$

where this semi-direct product corresponds to the natural action of $\pi_1(Y, T)$ on $\lim_{\rightarrow}(G, \widetilde{Y})$.

To see this recall the canonical surjection

$$\pi_1(G, Y, T) \to \pi_1(Y, T),$$

which is defined by simply killing elements of the local groups. If we denote this map by ρ and let $K = \operatorname{Ker} \rho$ then we have the short exact sequence,

$$1 \to K \to \pi_1(G, Y, T) \xrightarrow{\rho} \pi_1(Y, T) \to 1,$$

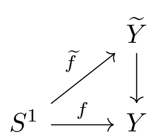
which splits since $\pi_1(Y, T)$ is a free group. That is,

$$\pi_1(G, Y, T) \cong K \rtimes \pi_1(Y, T).$$

Hence we need only show that K is canonically isomorphic to $\pi_1(G, \widetilde{Y}, \widetilde{Y})$. For this, thinking of $\pi_1(G, Y, T)$ via the definition given via a basepoint, $g \in \pi_1(G, Y, T)$ should be thought of as a homotopy class

$$g = [g_0 y_1 g_1 \cdots g_{n-1} y_n g_n],$$

where $(y_1, \ldots, y_n)$ is some loop in Y and each $g_i \in G_{o(y_{i+1}}$. Since $\rho([g_0 y_1 g_1 \ldots g_{n-1} y_n g_n]) = [y_1 \ldots y_n]$ we see that $g \in K$ exactly when the path $(y_1, \ldots, y_n)$ is null-homotopic in Y. If $f : S^1 \to Y$ denotes this loop, being null-homotopic says there is a (unique, since Y is connected) lift of f to a loop in $\widetilde{Y}$, which we denote by $\widetilde{f}$ and whose image we denote $(\widetilde{y_1}, \ldots, \widetilde{y_n})$.



The desired isomorphism $K \cong \pi_1(G, \widetilde{Y}, \widetilde{Y})$ is then defined via the map

$$[g_0 y_1 g_1 \dots g_{n-1} y_n g_n] \mapsto [g_0 \widetilde{y_1} g_1 \dots g_{n-1} \widetilde{y_n} g_n].$$

Now we turn our attention to reduced words in $F(G, Y)$ which is a technical notion which will be necessary in proving the structure theorem that is the goal of this chapter. Throughout Y is still a connected non-empty graph. Let (c, μ) be a word of type c where c is a path with origin P_0 and edges $y_1, \dots, y_n$. Also set $\mu = (r_0, \dots, r_n)$. We say that (c, μ) is *reduced* if either $n = 0$ and $r_0 \neq 1$ or if $n \geq 1$ then $r_i \notin G_{y_i}^{y_i}$ for any index i such that $y_{i+1} = \overline{y_i}$. In particular if c is a path without backtracking then (c, μ) is reduced. The main theorem regarding reduced words is the following.

Theorem 5.2. *If (c, μ) is a reduced word then the associated element $|c, \mu|$ of $F(G, Y)$ is not* 1.

The method of proof will be to first assume the result holds on small subgraphs of Y and use this to conclude the general result. The proof will then be complete when we provide the argument that the result does indeed hold on the necessary subgraphs. As such let us first discuss some corollaries of the theorem which will be useful in its proof.

Corollary 5.3. *The homomorphisms $G_P \to F(G, Y)$ are injective.*

Note this follows simply from the case of Theorem 5.2 where c has length 0.

Corollary 5.4. *If (c, μ) is reduced and $\ell(c) \geq 1$ then $|c, \mu| \notin G_{P_0}$ where P_0 is the origin of c.*

Proof. If $|c, \mu| = x \in G_{P_0}$ we would then have the reduced word (c, μ') given by

$$\mu' = (x^{-1} r_0, r_1, \dots, r_n),$$

and $|c, \mu'| = 1$ contradciting the theorem. □

Corollary 5.5. *For T a maximal tree of Y and (c, μ) a reduced word with c a closed path in Y, one has $|c, \mu| \neq 1$ in $\pi_1(G, Y, T)$.*

Proof. For $P_0 = o(c)$, the theorem says $|c, \mu| \neq 1$ in $\pi_1(G, Y, P_0)$ and by the isomorphism in Proposition 5.1 we have that $|c, \mu| \neq 1$ in $\pi_1(G, Y, T)$. □

Now to build up to the proof of the theorem. We aim ultimately to reduce ourself to the case where Y is either a segment or a loop. To do this suppose Y' is a connected non-empty subgraph of Y, with corresponding graph of groups $(G|_{Y'}, Y')$ and assume that the theorem holds for $(G|_{Y'}, Y')$. By Corollary 5.3 the maps $G_P \to F(G|_{Y'}, Y')$ are injective for all $P \in \operatorname{vert} Y'$. Let $W = Y/Y'$ denote the graph obtained from Y by contracting Y' to a point and denote this point by (Y') in W. Then we know

$$\operatorname{vert} W = (\operatorname{vert} Y \setminus \operatorname{vert} Y') \cup \{(Y')\},$$
$$\operatorname{edge} W = \operatorname{edge} Y \setminus \operatorname{edge} Y'.$$

We now define a graph of groups (H, W) on W as follows: If $P \in \operatorname{vert} Y \setminus \operatorname{vert} Y'$ we set $H_P = G_P$ and if $P = (Y')$ we set $H_P = F(G|_{Y'}, Y')$. If $y \in \operatorname{edge} W$ we set $H_y = G_y$. Note that our remark that the maps $G_P \to F(G|_{Y'}, Y')$ are injective for all $P \in \operatorname{vert} Y'$ assures that the map $H_y \to F(G|_{Y'}, Y')$ is injective for any edge with (Y') as an extremity. As constructed the natural projection map

$(G,Y) \to (H,W)$ induces a homomorphism $F(G,Y) \to F(H,W)$ which is plainly seen to be an isomorphism as the later is simply obtained by first appending the relations in the subgraph Y' and then adding the relations in ther rest of Y.

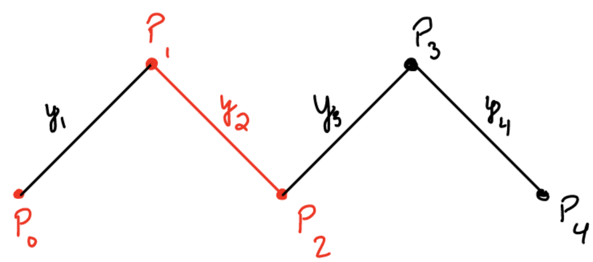
With (H,W) constructed and an isomorphism $F(G,Y) \to F(H,W)$ established we now want to construct for each word (c,μ) in $F(G,Y)$ a corresponding word (c',μ') in $F(H,W)$ so that $|c',\mu'|$ is the image of $|c,\mu|$ under the isomorphism. To this end, let $(P_0,\dots,P_n)$ be the sequence of vertices of c, $(y_1,\dots,y_n)$ the sequence of edges. Let $\mu = (r_0,\dots,r_n)$. For $1 \leq i \leq j \leq n$ we let (c_{ij},μ_{ij}) denote the subword corresponding the the subpath $(P_i,P_{i+1},\dots,P_j)$ and $\mu_{ij} = (r_i,\dots,r_j)$. If c_{ij} is entirely contained in Y' we let r_{ij} denote the element $r_i y_{i+1}\dots y_j r_j \in F(G|_{Y'},Y') = H_{(Y')}$. We then define and increasing sequence of integers

$$0 \leq i_0 \leq j_0 < i_1 \leq j_1 < \dots < i_m \leq j_m \leq n,$$

which break c up into segments such that each $c_{i_a j_a}$ is entirely contained in Y' and such that any vertex (resp. edge) which is contained in Y' belongs to on of the $c_{i_a j_a}$. In this way each intermediate path $c_{j_a i_{a+1}}$ is a path of length ≥ 1 with no vertices (except for its extremities) belonging to Y'. We set,

$$c' = (\dots, c_{j_{a-1} i_a}, c_{j_a i_{a+1}}, \dots),$$
$$\mu' = (\dots, \mu_{j_{a-1}+1, i_{a-1}}, r_{i_a j_a}, \mu_{j_a+1, i_{a+1}-1}, \dots).$$

As a clarifying example consider the following path of length 4 where the red vertices and edges are those belonging to Y'.



Then our sequence of integers is

$$0 \leq i_0 = 0 \leq j_0 = 0 < i_1 = 1 \leq j_1 = 2,$$

so that

$$c' = (y_1, y_3, y_4),$$

and

$$\mu' = (r_0, r_1 y_2 r_2, r_3, r_4).$$

The crucial lemma regarding (H,W) is the following.

Lemma 5.6. *If (c,μ) is a reduced word of (G,Y) then (c',μ') is a reduced word of (H,W).*

Proof. If $\ell(c') = 0$ and c' corresponds to the vertex $P \neq (Y')$ then it must hold that $\ell(c) = 0$ and $c = P$. Then (c,μ) being reduced says that $\mu = r_0 \in G_P$ and $r_0 \neq 1$ and since $G_P = H_P$ we have that (c',μ') is also reduced. If $P = (Y')$ we have that c is entirely containted in Y' and since the theorem is assumed to hold in Y' we have that $|c,\mu| \neq 1$. Since $|c',\mu'|$ is the image of $|c,\mu|$ under the isomorphism $F(G,Y) \to F(H,W)$ it follows that $|c',\mu'| \neq 1$.

Now suppose $\ell(c') \geq 1$ and let $w_1, \ldots, w_p$ denote the edges of c'. Recall that we aim to show that if $w_{h+1} = \overline{w_h}$ then $r'_h \notin H_{w_h}^{w_h}$ where $\mu' = (r'_0, \ldots, r'_p)$. For such an h, let $P = t(w_h)$. If $P \neq (Y')$ then the result follows from (c, μ) being reduced. If $P = (Y')$ we distinguish two cases. First suppose (w_h, r'_h, w_{h+1}) is of the form (y_i, r_i, y_{i+1}) with i an index such that $y_{i+1} = \overline{y_i}$. In other words the backtracking in the path c' arose from backtracking in c. Then by assumption that (c, μ) is reduced we have that $r_i \notin G_{y_i}^{y_i}$ and r_h is the image of r_i in $H_{(Y')}$. Our hypothesis gives that $G_{t(y_i)} \to H_{(Y')}$ is injective and via the identification $G_{y_i} = H_{w_h}$ we have that this injection maps $G_{y_i}^{y_i}$ onto $H_{w_h}^{w_h}$. Thus since $r_i \notin G_{y_i}^{y_i}$ we have that $r_h \notin H_{w_h}^{w_h}$.

Otherwise suppose (w_h, r'_h, w_{h+1}) is of the form $(y_{i_a}, r_{i_a j_a}, y_{j_a+1})$ with $i_a < j_a$ (i.e. there was no backtracking in c, but instead in c at some point the path entered Y' and left along the same edge). By definition

$$r_{i_a j_a} = |c_{i_a j_a}, \mu_{i_a j_a}| \in F(G|_{Y'}, Y').$$

Now since $\ell(c_{i_a j_a}) \geq 1$ Corollary 5.4 gives that $r_{i_a j_a} \notin G_Q$ where $Q = o(c_{i_a j_a}) = t(y_{i_a})$. In particular $r_{i_a j_a} \notin H_{w_h}^{w_h}$ since this is a subgroup of G_Q. □

Combining the isomorphism $F(G, Y) \to F(H, W)$ with the previous lemma we obtain the following:

Lemma 5.7. *If Theorem 5.2 holds for (H, W) then it holds for (G, Y).*

Proof of Thm 5.2. First we consider the case where Y is a segment consisting of a single geometric edge $\{y, \overline{y}\}$ connecting points P_{-1} and P_1. Then an element $|c, \mu|$ corresponding to a reduced word (c, μ) is of the form,

$$r_0 y^{e_1} r_1 y^{e_2} \ldots y^{e_n} r_n,$$

where each $e_i \in \{\pm 1\}$, $e_{i+1} = -e_i$ and $r_0 \in P_{-e_1}, r_i \in G_{P_{e_i}} \setminus G_y^{y^{e_i}}$. If $n = 0$ the $r_0 \neq 1$. If $n \geq 1$ let ϕ denote the canonical homomorphism

$$F(G, Y) \to \pi_1(G, Y, Y) = G_{P_{-1}} *_{G_y} G_{P_1}.$$

Then $\phi(|c, \mu|) = r_0 r_1 \ldots r_n$ and by the structure theorem for amalgams this is not 1 and therefore $|c, \mu| \neq 1$.

Now consider more generally the case where Y is a tree. By passing to direct limits we can assume that Y is finite. We then induct on $n = \frac{1}{2}\,\mathrm{card}(\mathrm{edge}\, Y)$ with $n = 0$ being trivial. If $n \geq 1$ we take Y' to be a segment contained in Y. By the previous case, the theorem holds on Y'. On the other hand Y/Y' is still a tree with fewer edges and so by induction the theorem holds for (H, W) where $W = Y/Y'$. By Lemma 5.7 this completes the proof.

Next consider the case where Y is a loop with a single vertex 0 and a single geometric edge $\{y, \overline{y}\}$. As previously discussed $F(G, Y)$ is an HNN extension,

$$F(G, Y) = \langle \mathscr{S}_0, y, | \mathscr{R}_0, y a^y y^{-1} = a^{\overline{y}} \rangle.$$

Setting $G_n = y^n G_0 y^{-n}$ and $A = G_y$, the group R defined to be the smallest normal subgroup containing G_0 is the sum of G_n amalgamated over the maps

$$G_{n-1} \leftarrow A \to G_n,$$
$$y^{n-1} a^{\overline{y}} y^{1-n} \leftarrow\!\shortmid a \mapsto y^n a^y y^{-n}.$$

Then a reduced word (c, μ) gives rise to an element $|c, \mu|$

$$r_0 y^{e_1} r_1 y^{e_2} \ldots y^{e_n} r_n,$$

where $r_i \in G_0$, $e_i \in \{\pm 1\}$ and $r_i \notin A^{y^{e_i}}$ if $e_{i+1} = -e_i$. Note if $\sum e_i \neq 0$, then $|c, \mu| \notin R$ and so $|c, \mu| \neq 1$. Thus we may assume $\sum e_i = 0$ and with this assumption we set

$$d_i = e_1 + \cdots + e_i, \quad s_i = y^{d_i} r_i y^{-d_i}.$$

Then

$$|c, \mu| = s_0 s_1 \ldots s_n$$

with each $s_i \in G_{d_i}$, $d_0 = d_n = 0$, $d_{i+1} - d_i = e_{i+1} = \pm 1$, and $s_i \notin y^{d_i} A^{y^{e_i}} y^{-d_i}$ if $d_{i+1} = d_{i-1}$. We let T be the tree whose vertices are indexed by $\mathbb{N}$ with unique edges connecting n to $n+1$. We then form the tree of groups (K, T) by assigning the group G_n to the vertex n and A to each edge with the injections being our maps $G_{n-1} \leftarrow A \rightarrow G_n$. As we noted when we defined R, it then follows that $R = \pi_1(K, T, T)$ and $s_0 s_1 \ldots s_n$ is associated to some reduced word of (K, T) which corresponds to a closed path in T since $d_0 = d_n = 0$. But by Corollary 5.5 this implies that $s_0 s_1 \ldots s_n = |c, \mu|$ is not 1 in $\pi_1(K, T, T) = R \leq F(G, Y)$, proving the result in this case.

We now have the tools to prove the completely general case. As in the case where Y was an arbitrary tree, by passing to direect limits we may assume that Y is finite. Again, we argue by induction on $n = \frac{1}{2}\operatorname{card}(\operatorname{edge} Y)$ with the case $n = 0$ being trivial. With $n \geq 1$ we then choose a subgraph of Y' with two edges, so either a segment or a loop. In either case we know Theorem 5.2 applies to Y' and via induction we get that Theorem 5.2 applies to Y/Y' and thus Lemma 5.7 yields the result. □

5.2. **Universal Covering of a Graph of Groups.** The last tool needed before stating and proving the structure theorem for groups acting on trees is the notion of the universal cover of a graph of groups. As setup we take (G, Y) a connected, non-empty graph of groups, T a maximal tree of Y and A an orientation of Y. Throughout for $y \in \operatorname{edge} Y$ we let,

$$e(y) = \begin{cases} 0 & y \in A, \\ 1 & y \notin A \end{cases}.$$

Now we aim to construct the following:

- A *tree* $\widetilde{X} = \widetilde{X}(G, Y, T)$,
- An action of $\pi := \pi_1(G, Y, T)$ on $\widetilde{X}$,
- A morphism $p : \widetilde{X} \rightarrow Y$ inducing an isomorphism $\pi \backslash \widetilde{X} \rightarrow Y$ (i.e. we aim to have that the quotient of $\widetilde{X}$ under the π action is precisely Y),
- Sections $\operatorname{vert} Y \rightarrow \operatorname{vert} \widetilde{X}, \operatorname{edge} Y \rightarrow \operatorname{edge} \widetilde{X}$ which we denote by $P \mapsto \widetilde{P}, y \mapsto \widetilde{y}$ such that,
- Under the π action the stabilizer of the point $\widetilde{P}$, $\pi_{\widetilde{P}}$ is precisely G_P and the stabilizer of $\widetilde{y}$, $\pi_{\widetilde{y}}$ is precisely the subgroup G_y viewed as a subgroup of $G_{o(y)}$ if $y \in A$ and viewed as a subgroup of $G_{t(y)}$ if $y \notin A$. Recall that these are indeed subgroups of π via Corollary 5.3.

It then immediately follows that

$$\operatorname{vert} \widetilde{X} = \bigsqcup_{P \in \operatorname{vert} Y} \pi / \pi_{\widetilde{P}}, \quad \operatorname{edge} \widetilde{X} = \bigsqcup_{y \in \operatorname{edge} Y} \pi / \pi_{\widetilde{Y y}}.$$

Before defining the endpoints of edges, we will let the section $\widetilde{P}$ and $\widetilde{y}$ simply be the coset represented by 1 in $\pi/\pi_{\widetilde{P}}$ and $\pi/\pi_{\widetilde{y}}$ respectively. Then every point and every edge in $\widetilde{X}$ is simply a π translate of some vertex or edge which was lifted from Y. Of course since there could be loops in Y and we wish for $\widetilde{X}$ to be a tree, the matter of lifting edges is a bit more delicate than simply lifting an edge to have endpoints the lift of its endpoints in Y. For $\widetilde{y} \in \text{edge}\,\widetilde{X}$ and $g \in \pi$ we propose the following deifnitions:

$$\overline{g\widetilde{y}} = g\overline{\widetilde{y}},$$
$$o(g\widetilde{y}) = gg_y^{e(y)}\widetilde{o(y)},$$
$$t(g\widetilde{y}) = gg_y^{1-e(y)}\widetilde{t(y)},$$

What we are doing here is lifting edges in T to the obvious edges in $\widetilde{X}$, but to avoiding lifting loops to loops, for $y \notin T$ we lift y to an edge which has origin in the lift of T if $y \in A$ and which has terminus in the lift of T if $y \notin A$. Then the rest of the graph's edges are determined by the π action on the "lift" of Y. Now we check that this definition of edges yields the desired edge stabilizers. I.e. we check that if $h \in \pi_{\widetilde{y}}$ then indeed h fixes both $o(\widetilde{y})$ and $t(\widetilde{y})$. By definition the stabilizer of $\widetilde{y}$ is the intersection of the stabilizers of its endpoints in π. Thus if $y \in A \setminus T$ we have that

$$\pi_{\widetilde{y}} = G_{o(y)} \cap g_y G_{t(y)} g_y^{-1} \supseteq G_{o(y)} \cap g_y G_y^y g_y^{-1} = G_{o(y)} \cap G_{\overline{y}}^{\overline{y}} = G_{\overline{y}}^{\overline{y}}.$$