REAL ANALYSIS LECTURE NOTES

ERIC ALBERS

Contents

These notes are based off a first-year graduate course in Real Analysis using the Folland's textbook.

1. Set Theoretic Preliminaries

This section is a brief summary of important content which can be found in chapter 0 from Folland's book. For additional details and proofs of theorems see the text. Given a—potentially uncountably infinite—family of sets indexed by A, $\{E_i\}_{\alpha\in A}$, we say this collection is *disjoint* if the elements are pairwise disjoint. That is, if $\alpha, \beta \in A$ with $\alpha \neq \beta$, then $E_\alpha \cap E_\beta = \varnothing$. The *limit superior* of a collection is given by

$$\limsup E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n,$$

and the *limit inferior* by

$$\liminf E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n.$$

Given two non-empty sets X and Y we define the following

$$\operatorname{card} X \leq \operatorname{card} Y, \quad \operatorname{card} X = \operatorname{card} Y, \quad \operatorname{card} X \geq Y,$$

to mean that there exists a injective, bijective, surjective $f : X \to Y$ respectively. We present the following propositions and theorems without proof but invite the reader to seek out pages 7-9 of the text to find the proofs.

Proposition 1.1. $\operatorname{card} X \leq \operatorname{card} Y$ *if and only if* $\operatorname{card} Y \geq \operatorname{card} X$.

Proposition 1.2. *For any two sets X, Y either* $\operatorname{card} X \leq Y$ *or* $\operatorname{card} Y \leq X$.

Theorem 1.3. *If* $\operatorname{card} X \leq \operatorname{card} Y$ *and* $\operatorname{card} Y \leq \operatorname{card} X$*, then* $\operatorname{card} X = \operatorname{card} Y$.

Proposition 1.4. *For any set X, $X < \mathscr{P}(X)$.*

We say a set is *countable* if $\operatorname{card} X \leq \operatorname{card} \mathbb{N}$ and *countably infinite* if equality holds.

Proposition 1.5.

- *If X and Y are countable, so too is $X \times Y$.*
- *If A is countable and all X_α are countable for $\alpha \in A$, then $\cup_{\alpha \in A} X_\alpha$ is countable.*

2. Measures

2.1. **σ-Algebras.** This section closely mirrors chapter 1 in Folland's book. In general, a motivating goal for this course is to develop a notion of size for *nice-enough* sets which will commonly be subsets of $\mathbb{R}^n$ for some n. We will denote this size map $m : X \to \mathbb{R}_{\geq 0}$, where X our space. Ideally we wish for m to satisfy the following:

(1) $m([0,1]^d) = 1$,
(2) $m(A \bigsqcup B) = m(A) + m(B)$,
(3) If $A_1 \subseteq A_2 \subseteq \ldots$ then as a sequence in $\mathbb{R}$ $m(A_n) \to m(A)$ where

$$A := \bigcup_{n=1}^{\infty} A_n.$$

(4) Given $T \in \operatorname{Isom}(\mathbb{R}^d)$ we have $m(A) = m(T(A))$.

Moreover we wish to develop this notion to be able to handle more complicated sets than sets which look like the geometric objects one studies in high school.

With this in mind, one might ask if it is possible to assign a notion of size to all subsets of $\mathbb{R}^d$. Sadly, the answer is no. To see this, consider the following equivalence relation $\sim$ on $[0,1]$ where $x \sim y$ if $x - y \in \mathbb{Q}$, and let E be a complete set of representatives of equivalence classes of $[0,1]/\sim$. Note that for all $q_1, q_2 \in \mathbb{Q} \cap [-1,1]$ we have that $E + q_1$ is disjoint from $E + q_2$. Since

$$[0,1] \subseteq \bigsqcup_{q \in Q \cap [-1,1]} (E + q) \subseteq [-1,2].$$

But since simply adding q is a isometry of $\mathbb{R}$ we should have $m(E+q)=m(E)$ for all $q\in\mathbb{Q}\cap[-1,1]$ and hence if $m(E)\neq 0$ the size of the disjoint union is necesarily infinite by requirements 2 and 3 above. Whether $m(E)=0$ or the disjoint union has infinite size clearly contradicts the above containments.

This leads to a necessary discussion on which sets such a function may exist. For that we need the notion of a *σ-algebra*. Given a space of X we say an *algebra of sets* is a family $\mathscr{M}\subseteq\mathscr{P}(X)$ which is closed under finite unions and taking complements. We call an algebra a *σ-algebra* if it is closed under countable unions. Note that from this definition we immediately have $\varnothing, X\in\mathscr{M}$ and $\mathscr{M}$ is closed under finite intersections if it is an algebra—countable intersections if it is a σ-algebra. From this it immediately follows that $\mathscr{M}$ is also closed under differences since $A\setminus B = A\cap B^c$.

For some examples, we have the trivial σ-algebra which consists only of $\varnothing$ and X for any set X. Also, $\mathscr{P}(X)$ is trivially a σ-algebra for any set X. For the first non-remedial example, for an uncountable set X, note that

$$\mathscr{M}=\{A\subseteq X|A \text{ is countable or co-countable }\},$$

forms a σ-algebra.

For our final example, consider the collection

$$\mathscr{M}=\{\text{ finite unions of intervals in } \mathbb{R}\}.$$

We claim this set is an algebra and to prove this we require the folllowing definition and lemma. We say $\mathscr{E}\subseteq\mathscr{P}(X)$ is an *elementary family* if the following conditions hold:

- $\varnothing\in\mathscr{E}$,
- If $A,B\in\mathscr{E}$, then $A\cap B\in\mathscr{E}$,
- If $A\in\mathscr{E}$, then A^c is a finite disjiont union of elements of $\mathscr{E}$.

{Lemma1.1}

Lemma 2.1. *If $\mathscr{E}$ is an elementary family then the collection*

$$\mathscr{M}=\{A=\bigsqcup_{i=1}^{k}E_i|E_i\in\mathscr{E}\},$$

is an algebra.

Proof. If $A,B\in\mathscr{E}$ and $B^c=\sqcup_{j=1}^{J}C_j$ where all $C_j\in\mathscr{E}$, then $A\setminus B=\bigsqcup_{j=1}^{J}(A\cap C_j)$ and $A\cup B=(A\setminus B)\sqcup B$. Hence $A\setminus B, A\cup B\in\mathscr{M}$ and the fact that $\mathscr{M}$ is closed under finite unions follows via induction.

For complements, let $A_1,\dots,A_n\in\mathscr{E}$ be a finite disjoint finite collection. Then write

$$A_m^c=\bigcup_{j=1}^{J_m}B_m^j,$$

where each $B_m^1,\dots,B_m^{J_m}$. Then

$$\left(\bigsqcup_{m=1}^{n}A_m\right)^c=\bigcap_{m=1}^{n}\left(\bigcup_{j=1}^{J_m}B_m^j\right)=\bigcup\{\cap B_m^{j_m}|1\leq m\leq n, 1\leq j_m\leq J_m\},$$

which is an element of $\mathscr{M}$ since finite intersections belong to $\mathscr{E}$ and we already showed $\mathscr{M}$ is closed under finite unions. $\square$

Given a family of algebra—or σ-algebra—denoted $\{\mathcal{A}_i\}_{i\in I}$ it is plain to see that

$$\bigcap_{i\in I}\mathcal{A}_i,$$

is an algebra or σ-algebra repsectively. This leads to the following definition. Given any collection of sets $\mathcal{E}\subseteq\mathcal{P}(X)$, there exists a unique smallest algebra containing $\mathcal{E}$, namely

$$\mathcal{A}(\mathcal{E})=\bigcap_{\substack{\mathcal{E}\subseteq\mathcal{A}\\ \mathcal{A}\text{ an alg.}}}\mathcal{A}$$

This is the *algebra generated by* $\mathcal{E}$. Likewise the minimal σ-algebra generated by $\mathcal{E}$ is denoted $\mathcal{M}(\mathcal{E})$ and is also the intersection of all σ-algebra containing $\mathcal{E}$.

{Lem1.2}

Lemma 2.2. *Fix a space X and let $\mathcal{E},\mathcal{F}\subseteq\mathcal{P}(X)$. If $\mathcal{E}\subseteq\mathcal{M}(\mathcal{F})$ then $\mathcal{M}(\mathcal{E})\subseteq\mathcal{M}(\mathcal{F})$.*

Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of non-empty sets $X=\prod_{\alpha\in A}X_\alpha$ and π_α denote the projection maps. Given a σ-algebra $\mathcal{M}_\alpha$ on each X_α we define the *product σ-algebra* to be the σ-algebra generated by

$$\{\pi_\alpha^{-1}(E_\alpha)|E_\alpha\in\mathcal{M}_\alpha\},$$

and we denote this algebra via $\bigotimes_{\alpha\in A}\mathcal{M}_\alpha$. In the case where one has a countable collection of σ-algebras one quickly verifies that $\bigotimes_{i=1}^\infty\mathcal{M}_\rangle$ is generated by $\prod_{i=1}^\infty E_i$ for $E_i\in\mathcal{M}_\rangle$.

Given any topological space X we define the *Borel* σ-algebra on X to be the algebra generated by all open sets in X and denote it by $\mathcal{B}_X$. On homework 1 the following result was proved for the Boreal algebra on $\mathbb{R}$ using Lemma 2.2

{Prop1.3}

Proposition 2.3. *$\mathcal{B}_\mathbb{R}$ is generated by any of the following collections.*

a) the open intervals $\mathcal{E}_1=\{(a,b)|a<b\}$,
b) the closed intervals $\mathcal{E}_2=\{[a,b]|a<b\}$,
c) the half-open intervals $\mathcal{E}_3=\{(a,b]|a<b\}$ or $\mathcal{E}_4=\{[a,b)|a<b\}$,
d) the open rays $\mathcal{E}_5=\{(a,\infty)|a\in\mathbb{R}\}$ or $\mathcal{E}_6=\{(-\infty,a)|a\in\mathbb{R}\}$,
e) the closed rays $\mathcal{E}_7=\{[a,\infty)|a\in\mathbb{R}\}$or $\mathcal{E}_8=\{(-\infty,a]|a\in\mathbb{R}\}$.

We now discuss a manner in which one can build-up the Borel algebra for a topological space X. Clearly $\mathcal{B}_X$ contains all open and closed sets. Since the collection of open sets is contained under arbitrary unions, countable unions of open sets contribute nothing more to the algebra. Closed sets, however, are not closed under countable unions and hence we must add all sets which can be obtained as a countable union of closed sets in X. Such sets are referred to as F_σ sets. Taking complemenets we obtain countable intersections of open sets, which are usually referred to as G_δ sets. Continuining in this fashion, countable unions of G_δ sets are said to be of type $G_{\delta,\sigma}$ while countable intersections of F_σ are said to be of type $F_{\sigma,\delta}$. Iterating through this process indefinitely obtains all sets in the Borel algebra.

2.2. **Measures.** Let X be a space, $\mathcal{M}\subseteq\mathcal{P}(X)$ be a σ-algebra. A function $\mu:\mathcal{M}\to[0,\infty]$ is a *measure* if

i) $\mu(\varnothing)=0$,
ii) $\mu(\bigsqcup_{i=1}^\infty A_i)=\sum_{i=1}^\infty\mu(A_i)$.

For some examples first note that given any σ-algebra the function $\mu(\varnothing) = 0, \mu(A) = \infty$ for any $A \neq \varnothing$ defines a measure. We also have the counting measure on $\mathcal{P}(X)$ for any space X, given by

$$\mu(A) = \begin{cases} |A| & A \text{ finite}, \\ \infty & \text{otherwise}. \end{cases}$$

Given a space X and $x_0 \in X$ we have the Dirac measure δ_{x_0} on $\mathcal{P}(X)$ defined by

$$\delta_{x_0}(A) = \begin{cases} 1 & x_0 \in A, \\ 0 & \text{otherwise}. \end{cases}$$

These measures have some use, although the primary purpose for the next few sections of these notes will be to construct the Lebesgue measure m, on $\mathcal{B}_{\mathbb{R}}$ which satisfies $m([a, b]) = b - a$.

{Thm1.4}

Theorem 2.4. *Given a measure space $(X, \mathcal{M}, \mu)$ we have the following.*

a) Given $E_1, \dots, E_n \in \mathcal{M}$ one has

$$\mu(E_1 + \cdots + E_n) = \mu(E_1) + \dots \mu(E_n).$$

b) If $E, F \in \mathcal{M}$ are such that $E \subseteq F$ then $\mu(E) \leq \mu(F)$.

c) Given $\{E_i\}_1^\infty$ we have $\mu(\cup_1^\infty E_i) \leq \sum_{i=1}^\infty E_i$.

d) If $E_1 \subseteq E_2 \subseteq \dots$ is an increasing sequence of elements in $\mathcal{M}$ then

$$\mu\left(\bigcup_{i=1}^\infty E_i\right) = \lim_{n\to\infty} \mu(E_n).$$

e) If $E_1 \supseteq E_2 \supseteq \dots$ is a decreasing sequence of elements in $\mathcal{M}$ and $\mu(E_1) < \infty$ then

$$\mu\left(\bigcap_{i=1}^\infty E_i\right) = \lim_{n\to\infty} \mu(E_n).$$

Proof. First, μ is clearly finitely additive, for given a finite collection $E_1, \dots, E_n \in \mathcal{M}$, the infinite sequence $E_1, \dots, E_n, \varnothing, \dots, \varnothing$ together with the axiom of countable additivity gives the result.

To see *b)* simply observe,

$$\mu(F) = \mu(F \setminus E \sqcup E) = \mu(F \setminus E) + \mu(E) \geq \mu(E).$$

For *c)* define for each k

$$F_k := E_k \setminus \bigcup_{i=1}^{k-1} E_i.$$

Then we have $\cup_1^\infty F_k = \cup_1^\infty E_k$ giving

$$\mu\left(\bigcup_{i=1}^\infty F_i\right) = \mu\left(\bigcup_{i=1}^\infty E_i\right).$$

Since the F_i's are disjoint and $F_k \subseteq E_k$ for each k we conclude

$$\mu\left(\bigcup_{i=1}^\infty E_i\right) = \mu\left(\bigcup_{i=1}^\infty F_i\right) = \sum_{i=1}^\infty \mu(F_i) \leq \sum_{i=1}^\infty \mu(E_i),$$

using *b)*.

For *d)* we note that $E_j \setminus E_{j-1}$—defining $E_0 := \varnothing$—is a disjoint sequence of sets since the E_i are increasing. From this we obtain

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} E_i \setminus E_{i-1}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i \setminus E_{i-1}) = \lim_{n\to\infty} \mu(E_n).$$

Finally, for *e)* define

$$F_k := E_1 \setminus E_k,$$

for each k. Then $F_1 \subseteq F_2 \subseteq \dots$, $\mu(E_1) = \mu(F_k) + \mu(E_k)$ for each k and

$$\bigcup_{i=1}^{\infty} F_k = E_1 \setminus \left(\bigcap_{i=1}^{\infty} E_j\right).$$

By *d)* then

$$\mu(E_1) = \mu\left(\bigcap_{i=1}^{\infty} E_i\right) + \lim_{n\to\infty} \mu(F_n) = \mu\left(\bigcap_{i=1}^{\infty} E_i\right) + \lim_{n\to\infty} [\mu(E_1) - \mu(E_n)].$$

Since $\mu(E_1) < \infty$ we may subtract it from the left-most and right-most sides above to achieve the desired result. □

In any measure space if $\mu(E) = 0$ monotonicity guarantees that $\mu(F) = 0$ for any $F \subseteq E$. A measure space $(X, \mathcal{M}, \mu)$ such that for every $E \in \mathcal{M}$ satisfying $\mu(E) = 0$ we have for each $F \subseteq E$, $F \in \mathcal{M}$ is said to be *complete*.

{Thm1.5}

Theorem 2.5. *Suppose $(X, \mathcal{M}, \mu)$ is a measure space and define the collections*

$$\mathcal{N} = \{N \in \mathcal{M} | \mu(N) = 0\}, \quad \overline{\mathcal{M}} = \{E \cup F | E \in \mathcal{M} \text{ and } F \subseteq N \text{ for some } N \in \mathcal{N}\}.$$

Then $\overline{\mathcal{M}}$ is a σ-algebra and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$.

Proof. Begin by noting that since both $\mathcal{M}, \mathcal{N}$ are closed under countable unions—the second due to σ-sub additivity—we obtain that $\overline{\mathcal{M}}$ is thus closed under countable unions. For complements let $E \cup F \in \overline{\mathcal{M}}$ where $F \subseteq N$ for some $N \in \mathcal{M}$ of 0 measure. We may assume $E \cap N = \varnothing$ since otherwise we could just rewrite $E \cup F = E \cup (F \setminus E)$ where $F \setminus E \subseteq N \setminus E$. Under this assumption we may then write

$$E \cup F = (E \cup N) \cap (N^c \cup F),$$

from which we obtain

$$(E \cup F)^c = (E \cup N)^c \cup (N \setminus F),$$

where the left term above belongs to $\mathcal{M}$ and right term is a subset of N. Thus $\overline{\mathcal{M}}$ is a σ-algebra.

We leave it as an exercise to verifty that the measure $\overline{\mu}(E \cup F) = \mu(E)$ is the unique extension of μ to $\overline{\mathcal{M}}$ and that this measure is complete. □

2.3. **Outer Measures.** Given a set X an *outer measure* is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that the following conditions holds:

- $\mu^*(\varnothing) = 0$,
- If $A \subseteq B$ then $\mu^*(A) \le \mu^*(B)$,

- For any countable collection $A_1, A_2, \dots$, one has

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i).$$

The following lemma provides with a concrete method of constructing an outer measure.

{Lem1.6}

Lemma 2.6. *Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a subcollection of subsets of X containing $\varnothing$ and X and let $\rho : \mathcal{E} \to [0, \infty]$ be any function with $\rho(\varnothing) = 0$. Then the following defines an outer measure on X.*

$$\mu^*(A) = \inf\left\{\sum_{i=1}^{\infty} \rho(E_i) | A \subseteq \bigcup_{i=1}^{\infty} E_i\right\}.$$

Proof. Trivially $\mu^*(\varnothing) = 0$ by covering $\varnothing$ by countable copies of the empty set. For monotonicity suppose $A \subseteq B$ note that any cover of B is a cover of A. Thus $\mu^*(A)$ for the collection

$$\left\{\sum_{i=1}^{\infty} \rho(E_i) | B \subseteq \bigcup_{i=1}^{\infty} E_i\right\},$$

and so upon taking infimums we see $\mu^*(A) \leq \mu^*(B)$. For σ-subadditivity let $A_1, A_2, \dots$ be a countable collection of subsets of X and $\epsilon > 0$. For each i there exists some countable cover $E_{i,j}$ such that

$$\mu^*(A_i) + \epsilon/2^i \geq \sum_{j=1}^{\infty} \rho(E_{i,j}).$$

From these cover we then obtain a cover for the infinite union, namely

$$\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty} E_{i,j}.$$

The proof is then complete upon noting the following chain of inequalities:

$$\begin{aligned}
\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \rho(E_{i,j}),\\
&\leq \sum_{i=1}^{\infty} (\mu^*(A_i) + \epsilon/2_i),\\
&\leq \left(\sum_{i=1}^{\infty} \mu^*(A_i)\right) + \epsilon.
\end{aligned}$$

□

Given an outer measure μ^* on X we define a set $A \subseteq X$ to be μ^*-*measureable* if for any $E \subseteq X$ one has

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

By finite sub-additivity we note the inequality $\leq$ above must hold and hence for all intents and purposes one only ever need show $\geq$ above. Moreover if $\mu^*(E) = \infty$ there is then nothing left to show and hence this need only be shown for sets of finite measure.

{Thm1.7}

Theorem 2.7. *Given a set X together with an outer measure μ^*, the collection of all μ^* measurable sets, denoted $\mathcal{M}$, forms a σ-algebra. The restriction $\mu^*|_{\mathcal{M}}$ is then a complete measure on $\mathcal{M}$.*

Proof. Closure under complements is trivial since the definition of μ^*-measurability is symmetric in A and A^c. We now show $\mathcal{M}$ is closed under finite unions. To that end, let $A, B \in \mathcal{M}$ and note for any $E \subseteq X$ by definition of A, B

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c),\\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu(E \cap A^c \cap B^c).\end{aligned}$$

Upon noting

$$E \cap (A \cup B) = (E \cap A \cap B) \cup (E \cap A \cap B^c) \cup (E \cap A^c \cap B),$$

finite sub-additivity of μ^* gives

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c),$$

which by previous marks is the only necessary inequality to conclude equality. Hence $\mathcal{M}$ forms an algebra.

Next we claim the restriction $\mu^*|_{\mathcal{M}}$ is finitely additive. To that end simply observe

$$\mu^*(A \sqcup B) = \mu^*((A \sqcup B) \cap A) + \mu^*((A \sqcup B) \cap A^c).$$

To conclude that $\mathcal{M}$ is in fact a σ-algebra we need only show that $\mathcal{M}$ is closed under countable disjoint unions. Letting $A_1, A_2, \dots \in \mathcal{M}$ be a countable disjoint collection for brevity we define

$$B := \bigsqcup_{i=1}^{\infty} A_i, \quad B_n := \bigsqcup_{i=1}^{n} A_i.$$

By closure under finite unions we note that each $B_n \in \mathcal{M}$. This means for any $n \in \mathbb{N}, E \subseteq X$, we have

$$\mu^*(E \cap B_n) + \mu^*(E \cap B_n^c).$$

Moreover

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c),\\ &= \mu * (E \cap A_n) + \mu^*(E \cap B_{n-1}),\end{aligned}$$

and thus via induction we obtain

$$\mu^*(E \cap B_n) = \sum_{i=1}^{n} \mu^*(E \cap A_i).$$

Thus for any E we have

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \geq \sum_{i=1}^{n} \mu^*(E \cap A_i) + \mu^*(E \cap B^c).$$

Letting $n \to \infty$ we obtain the following

$$\begin{aligned}\mu^*(E) &\geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap B^c), \\ &\geq \mu^*\left(\bigcup_{i=1}^{\infty}(E \cap A_i)\right) + \mu^*(E \cap B^c), \\ &= \mu^*(E \cap B) + \mu^*(E \cap B^c).\end{aligned}$$

This proves closure under countable unions and upon taking $E = B$ in the above we indeed have

$$\mu^*(B) = \sum_{i=1}^{\infty} \mu^*(A_i),$$

and so μ^* is countable additive on $\mathcal{M}$. For completeness simply note for $A \subset X$ such that $\mu^*(A) = 0$, for any $E \subseteq X$,

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu(E \cap A^c) \leq \mu^*(E).$$

Hence $A \in \mathcal{M}$ completing the proof. □

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra. Then $\rho : \mathcal{A} \to [0, \infty]$ is called a *premeasure* if $\rho(\varnothing) = 0$ and if $\{A_i\}_1^{\infty}$ is a collection of disjoint sets such that $A := \cup_1^{\infty} \in \mathcal{A}$ then $\rho(A) = \sum_{i=1}^{\infty} \rho(A_i)$.

{Prop1.8}

Proposition 2.8. *If ρ is a premeasure on $\mathcal{A}$ then defining the outer measure ρ^* as in Lemma 2.6 one has*

a) $\rho^*|_{\mathcal{A}} = \rho$,
b) Every set in $\mathcal{A}$ is ρ^*-measurable.

Proof. *a)* Let $E \in \mathcal{A}$ and suppose $E \subseteq \cup_1^{\infty} A_i$ for some $\{A_i\} \subseteq \mathcal{A}$. Defining

$$B_n := E \cap (A_n \setminus \bigcup_{i=1}^{n-1} A_i),$$

we see that the B_n all belong to $\mathcal{A}$ and are disjoint and hence

$$\rho(E) = \sum_{n=1}^{\infty} \rho(B_n) \leq \sum_{i=1}^{\infty} \rho(A_i),$$

where the inequality follows from monotonicity of a premeasure. Upon taking infimums of the above we obtain $\rho(E) \leq \rho^*(E)$. For the reverse inequality simply note the cover $\{A_i\}$ where $A_1 = E, A_n = \varnothing$ for $n \geq 2$ gives that $\rho^*(E) \leq \rho(E)$.

Let $A \in \mathcal{A}$, $E \subseteq X, \epsilon > 0$. Then there exists a sequence $\{B_j\}_1^{\infty} \subseteq \mathcal{A}$ with $E \subseteq \cup_1^{\infty} B_j$ such that

$$\sum_{j=1}^{\infty} \rho(B_j) \leq \rho^*(E) + \epsilon.$$

Since ρ is additive we obtain the following

$$\rho^*(E) + \epsilon \geq \sum_{j=1}^{\infty} \rho(B_j \cap A) + \sum_{j=1}^{\infty} \rho(B_j \cap A^c) \geq \rho^*(E \cap A) + \rho^*(E \cap A^c).$$

Taking $\epsilon \to 0$ we obtain the result. □

Before presenting the final, and most complete theorem for this section we make a brief remark out about sets of outer measure 0. Suppose μ^* is an outer measure on X and $A \subseteq X$ is such that $\mu^*(A) = 0$. Then A is μ^*-measurable. To see this, recall for μ^* measurability we need only show the inequality

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \setminus A).$$

The above follows immediately upon noting $\mu^*(E \cap A) = 0$ and $E \setminus A \subseteq E$. This is precisely how we showed that $\mu^*|_{\mathcal{M}}$ is a complete measure in Theorem 2.7.

{Thm1.9}

Theorem 2.9. *Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, μ_0 be a premeasure on $\mathcal{A}$ and $\mathcal{M}$ the σ-algebra generated by $\mathcal{A}$. There exists a measure μ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0. Moreover, if ν is another extension of μ_0 then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$ with equality if $\mu(E) < \infty$. This extension to a measure is unique if μ_0 is σ-finite.*

Proof. The first statement simply follows from Theorem 2.7 and Proposition 2.8. For the second claim let $E \in \mathcal{M}$ and suppose $E \subseteq \cup_1^\infty A_j$ for $A_j \in \mathcal{A}$. Then

$$\nu(E) \leq \sum_{j=1}^{\infty} \nu(A_j) = \sum_{j=1}^{\infty} \mu(A_j),$$

and so $\nu(E) \leq \mu(E)$ for any E. Setting $A := \cup_1^\infty A_j$, one has

$$\nu(A) = \lim_{n\to\infty} \nu\left(\bigcup_{j=1}^{n} Aj\right) = \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} Aj\right) = \mu(A).$$

If $\mu(E)$ is finite, then by definition of the infinite we may choose the A_j's so that $\mu(A) \leq < \mu(E) + \epsilon$ for any $\epsilon > 0$ and hence $\mu(A \setminus E) < \epsilon$. This gives

$$\mu(E) \leq \mu(A) = \nu(A) = \nu(E) + \nu(A \setminus E) \leq \nu(E) + \mu(A \setminus E) \leq \nu(E) + \epsilon,$$

and so $\mu(E) = \nu(E)$.

In the case where μ_0 is σ-finite we may write $X = \cup_1^\infty A_j$ where $\mu_0(A_j) < \infty$ for each A_j. We may assume without loss of generality that the A_j are disjoint so that

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E \cap A_j) = \sum_{j=1}^{\infty} \nu(E \cap A_j) = \nu(E),$$

for any $E \in \mathcal{M}$.

□

{Prop1.10}

2.4. Borel Measures on $\mathbb{R}$.

Proposition 2.10. *Let $\mathcal{A}$ be the algebra defined by generated by the half open intervals, $(a_i, b_i]$ for $a_i < b_i$ where we allow for the case where $a_i = -\infty$ or $b_i = \infty$. Then*

$$\mu_0\left(\bigsqcup_{i=1}^{k} (a_i, b_i]\right) = \sum_{i=1}^{k} (b_i - a_i),$$

defines a premeasure on $\mathcal{A}$.

Proof. We remark here that well-definedness of this map must be checked, but is a easy verification when dealing with finite unions so we omit the verification. Thus we need only check for $A \in \mathcal{A}$ such that $A = \sqcup_1^\infty A_i$ for $A_i \in \mathcal{A}$, one has

$$\mu_0(A) = \sum_{i=1}^{\infty} \mu_0(A_i).$$

We may reduce to the case where A is a single interval, namely $A = (a, b]$ and likewise each $A_i = (a_i, b_i]$. For any $n \in \mathbb{N}$ one has $\sqcup_1^n (a_i, b_i] \subseteq (a, b]$ and after renumerating our intervals so that $a \leq a_1 \leq b_1 \leq a_2 \leq \cdots < b$, from which we see

$$\mu_0(A) = b - a \geq \sum_{i=1}^{\infty} (b_i - a_i).$$

Thus

$$\mu_0(A) \geq \sum_{i=1}^{\infty} \mu_0(A_i).$$

For the other inequality let $\epsilon > 0$ and assume $-\infty < a, b < \infty$. Given $(a, b] \subseteq \sqcup_1^{\infty} (a_i, b_i]$ we then obtain the cover

$$[a + \epsilon, b] \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i + \epsilon 2^{-i}).$$

By compactness of $[a + \epsilon, b]$ the above admits a finite subcover, namely

$$[a + \epsilon, b] \subseteq \bigcup_{j=1}^{N} (a_{i_j}, b_{i_j} + \epsilon 2^{-i_j}).$$

Now having reduced ourselves to a finite cover, upon eliminating any intervals which are fully contained in another interval of the cover we may renumerate so that there exists i_j so that $a_{i_j} < a$, and an $i_{j'}$ such that $b_{i_{j'}} > b$ from which we see

$$b - a - \epsilon \leq \sum_{j=1}^{N} (b_{i_j} + \epsilon 2^{-i_j} - a_{i_j}).$$

From this it follows that

$$\begin{aligned} b - a = b - a - \epsilon + \epsilon &\leq \sum_{j=1}^{N} (b_{i_j} + \epsilon 2^{-i_j} - a_{i_j}) + \epsilon, \\ &\leq \sum_{i=1}^{\infty} (b_i + \epsilon 2^{-i} - a_i), \\ &= \sum_{i=1}^{\infty} (b_i - a_i) + 2\epsilon. \end{aligned}$$

Taking $\epsilon \to 0$ we obtain the result for finite a, b. In the case where $a = -\infty$ we may cover $[M, b]$ in the same manner and if $b = \infty$ we may cover $[a + \epsilon, M]$ in the same manner and taking $\epsilon \to 0, M \to \pm\infty$ we obtain the result for all half-open intervals. □

This proposition allows us to finally construct the *Lebesgue measure*. Extending μ_0 to an outer measure as in Lemma 2.6 we obtain a measure on the σ-algebra of outer-measurable sets which we denote by $\mathcal{L}$. Note since $\mathcal{B}_{\mathbb{R}}$ is generated by the half-open intervals we have $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{L}$. Henceforth we denote this measure by m. We now proceed through a few example calculations of the Lebesgue measure of sets. Let $\{x\} \subseteq \mathbb{R}$. Then upon covering x by the set $(x - 1/n, x + 1/n]$ monotonicty gives $m(x) \leq 2/n$ and taking $n \to 0$ we obtain $m(\{x\}) = 0$. From this we obtain the measure of any countable subset of $\mathbb{R}$ has measure 0 as it can be covered by the singletons of all its elements, all of which have measure 0.

We now provide a brief digression on the classical topic of rational approximation. By density of the rationals $\mathbb{Q} \subseteq \mathbb{R}$ given any $x \in \mathbb{R}, \epsilon > 0$, there exists $p, q \in \mathbb{N}$ such that $|x - p/q| < \epsilon$. It is a famous theorem of Dirichlet that for any $x \in \mathbb{R}, \epsilon > 0$ there exists infinitely $p, q \in \mathbb{N}$ such that

$$\left| x - \frac{p}{q} \right| < \frac{\epsilon}{q^2}.$$

As an exploration in the Lebesgue measure we now present and prove the following theorem.

{Thm1.11}

Theorem 2.11. *Fix $\epsilon > 0$. Then for Lebesgue almost every $x \in \mathbb{R}$ there are only finitely many $p, q \in \mathbb{Z}$ such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\epsilon}}.$$

Before proceeding with the proof we make some remarks on the limit superior of sets and their measures. Recall that given a collection of sets E_n its limit superior is given by

$$\limsup E_n = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} E_n = \{x \in X | x \in E_n \text{ for infinitely many } n\}.$$

We remark here that if $\sum_{k=1}^{\infty} \mu(E_k) < \infty$ then necessarily $\mu(\limsup E_n) = 0$. To see this, define $F_k := \cup_{n \geq k} E_n$ and note that the F_k then form a decreasing sequence whose limit is precisely the limsup and by sub-additivity one has

$$\mu(F_k) \leq \sum_{n=k}^{\infty} E_n.$$

From this we conclude

$$\mu(\limsup E_n) = \lim_{k \to \infty} \mu(F_k) \leq \lim_{k \to \infty} \sum_{n=k}^{\infty} \mu(E_n) = 0,$$

where the last equality follows since it is the tail of a convergent series. With this in mind we now present the proof of the theorem.

Proof of Theorem 2.11. We restrict ourselves to the interval $(0, 1)$ for simplicity but remark that the entire theorem immediately follows from this subcase. Let

$$A = \{x \in (0, 1) | \exists \text{ inf. many } p, q \in \mathbb{N}, p = 1, \ldots, q-1 \text{ s.t. } |x - p/q| < q^{-2-\epsilon}\}.$$

We aim to show A has Lebesgue measure 0. To that end, fixing $q \in \mathbb{N}, p = 1, \ldots, q-1$ we define

$$E_{p,q} := \{x \in (0,1) || x - p/q| < q^{-2-\epsilon}\} = \left(\frac{p}{q} - \frac{1}{q^{2+\epsilon}}, \frac{p}{q} + \frac{1}{q^{2+\epsilon}} \right),$$

and observe $m(E_{p,q}) = 2q^{-2-\epsilon}$. Upon noting that $A = \limsup E_{p,q}$, we need only show that $\sum_{p,q} m(E_{p,q}) < \infty$ to conclude the result from our previous remarks. To see this, simply note

$$\sum_{q=1}^{\infty} \sum_{p=1}^{q-1} \frac{2}{q^{2+\epsilon}} \leq \sum_{q=1}^{\infty} \frac{2q}{q^{2+\epsilon}} = \sum_{n=1}^{\infty} \frac{2}{n^{1+\epsilon}} < \infty,$$

as a p-series with $p > 1$. □

Before proving the so-called regularity properties of the Lebesgue measure, we present the following lemma, which says we may cover our sets with open intervals instead of left-open, right-closed intervals.

{Lem1.12}

Lemma 2.12. *Let $E \in \mathcal{L}_{\mathbb{R}}$. Then*

$$m(E) = \inf\left\{ \sum_{i=1}^{\infty}(b_i - a_i) \middle| E \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i) \right\}.$$

Proof. For brevity we will denote the quantity on the right by $\nu(E)$. Supposing $E \subseteq \cup_1^\infty (a_i, b_i)$ we may write each (a_i, b_i) as a countable disjoint union of left-open, right-closed intervals,

$$(a_i, b_i) = \bigsqcup_{j=1}^{k} I_i^j.$$

Then

$$\sum_{i=1}^{\infty}(b_i - a_i) = \sum_{i,j}^{\infty} \mu(I_i^j) \geq \mu(E).$$

Hence $\nu(E) \geq \mu(E)$.

For the other inequality. let $\epsilon > 0$. Then there exists a cover $E \subseteq \cup_1^\infty (a_i, b_i]$ such that

$$\sum_{i=1}^{\infty}(b_i - a_i) \leq \mu(E) + \epsilon.$$

Then for each i we consider the open interval $(a_i, b_i + \epsilon 2^{-i})$ so that $E \subseteq \cup_1^\infty (a_i, b_i + \epsilon 2^{-i})$. Upon noting

$$\nu(E) \leq \sum_{i=1}^{\infty}(b_i + 2\epsilon^{-i} - a_i) = \sum_{i=1}^{\infty}(b_i - a_i) + \epsilon \leq \mu(E) + 2\epsilon,$$

we obtain the other inequality taking $\epsilon \to 0$. □

{Thm1.13}

Theorem 2.13.

a) For every $E \in \mathcal{L}_{\mathbb{R}}$ and every $\epsilon > 0$ there exists an open $U \subseteq \mathbb{R}$ with $E \subseteq U$ and $m(U \setminus E) < \epsilon$.
b) For every $E \in \mathcal{L}_{\mathbb{R}}$ with $m(E) < \infty$ and for every $\epsilon > 0$ there exists a compact $K \subseteq E$ with $m(E \setminus K) < \epsilon$.
c) For every $E \in \mathcal{L}_{\mathbb{R}}$ with $m(E) < \infty$ and for every $\epsilon > 0$ there exists a finite union of disjoint open intervals with $m(A \Delta E) < \epsilon$.

Proof. For *a)* we first assume $m(E) < \infty$. In this case Lemma 2.12 gives an open cover $E \subseteq \cup_1^\infty (a_i, b_i)$ such that

$$\sum_{i=1}^{\infty}(b_i - a_i) \leq m(E) + \epsilon,$$

for any $\epsilon > 0$. Taking $U := \cup_1^\infty (a_i, b_i)$ the result is then obtained. In the case where $m(E) = \infty$ we simply split $\mathbb{R}$ into the disjoint intervals $I_n = (n, n+1]$ for each $n \in \mathbb{Z}$ and argue on each interval. In particular $E \cap I_n$ has a open set U such that $m(U_n) \leq m(E) + \epsilon 3 2^{-|n|}$. Then $\cup_{n \in \mathbb{Z}} U_n$ is the desired open set.

For *b)* suppose first that E is bounded, say $E \subseteq I$ for some closed interval I. Let $F = I \setminus E$. Then by the previous there exists an open $U \supseteq I \setminus E$ such that $m(U \setminus F) < \epsilon$. Defining $K = I \setminus U$ then $K \subseteq E$ and K is compact. Moreover,

$$m(E \setminus K) = m(U \setminus F) < \epsilon,$$

as desired. In the case where E is unbounded we simply divide $\mathbb{R}$ into intervals of finite measure as in the previous to conclude the result.

The proof of c is left as an exercise. □

{Cor1.14}

Corollary 2.14. *Let $E \in \mathcal{L}_{\mathbb{R}}$. Then $E = A \setminus N$ where $A \in G_\delta$—that is, A is a countable intersection of open sets—and $m(N) = 0$. Likewise $E = B \cup N'$ where $B \in F_\sigma$—that is, B is a countable union of closed sets—and $m(N') = 0$.*

We leave the proof of this corollary to the reader as it is a simple consequence of Theorem 2.13 The Lebesgue measure also exihibits the following geometric properties.

{Prop1.15}

Proposition 2.15. *Let $E \in \mathcal{L}_{\mathbb{R}}$. Then for any $x \in \mathbb{R}$, $m(E + x) = m(E)$ and $m(xE) = |x|m(E)$.*

The proof simply follows from definition of the measure and so we omit it.

To conclude this section, we discuss the more general *Lebesgue-Stieltjes Measures*. The construction is similar to the construction of the Lebesgue measure, namely through construction of a premeasure on the algebra of left-open, right-closed intervals.

{Prop1.16}

Proposition 2.16. *Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right-continuous. Then $\mu_0 : \mathcal{A} \to \mathbb{R}$ given by*

$$\mu_0\left(\bigsqcup_{i=1}^{k}(a_i, b_i]\right) = \sum_{i=1}^{k}(F(b) - F(a)),$$

defines a premeasure on $\mathcal{A}$ where $\mathcal{A}$ is the algebra of left-open, right-closed intervals.

We note here that $\mathcal{A}$ contains sets of the form $(-\infty, a], (a, \infty)$ and so we extend our definition on these intervals to be

$$\mu_0((\infty, a]) = F(b) - \lim_{t \to -\infty} F(t),$$
$$\mu_0((a, \infty)) = \lim_{t \to \infty} F(t) - F(a).$$

Proof of Proposition 2.16. It bears remarking that well-definedness of this function must be checked but we will not present this verification in the proof. Now let $A \in \mathcal{A}$ be such that $A = \sqcup_1^\infty A_i$ where each $A_i \in \mathcal{A}$. We may reduce to the case where $A = (a, b]$ and each A_i is likewise given by $A_i = (a_i, b_i]$. Then for any $n \in \mathbb{N}$,

$$\mu_0(A) = \mu_0\left(\bigcup_{i=1}^{n} A_i\right) + \mu_0\left(A \setminus \bigcup_{i=1}^{n} A_i\right) \geq \mu_0\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu_0(A_i),$$

giving $\mu_0(A) \geq \sum_{i=1}^{n} \mu_0(A_i)$.

For the reverse inequality first suppose a, b both finite and let $\epsilon > 0$. By right-continuity of F there exists a $\delta > 0$ such that $F(a + \delta) - F(a) < \epsilon$ and for each i there exists a δ_i such that $F(b_i + \delta_i) - F(a_i) < \epsilon 2^{-i}$. We then obtain the cover

$[a+\delta, b] \subseteq \cup_1^\infty (a_i, b_i + \delta_i)$ which being an open cover of a compact set admits a finite subcover. Denote this finite cover via

$$[a+\delta, b] \subseteq \bigcup_{j=1}^{N} (a_{i_j}, b_{i_j} + \delta_{i_j}),$$

and note upon reordering the intervals if necessary we have $a_{i_1} < a+\delta, b+\delta_{i_N} > b$. Then we have

$$\begin{aligned}\mu_0(I) = F(b) - F(a) &< F(b) - F(a+\delta) + \epsilon, \\ &\leq F(b+\delta_{i_N}) - F(a_1) + \epsilon, \\ &\leq F(b+\delta_{i_N}) - F(a_{i_N}) + \sum_{j=1}^{N-1} (F(b_{i_j} + \delta_{i_j}) - F(a_{i_j}) + \epsilon, \\ &< \sum_{i=1}^{\infty} (F(b_i) - F(a_i)) + 2\epsilon.\end{aligned}$$

This concludes the proof of the case where a, b are finite and we leave it to the reader to reconcile the other cases. □

3. Integration

3.1. **Measurable Functions.** We now use measures to develop a richer theory for integration than the one centered around the Riemann Integral. Given a measure space $(X, \mathcal{M}, \mu)$ and a function $f : X \to \mathbb{R}$ or $\mathbb{C}$ we aim to define a notion of some positive linear operator $\int_X d\mu$ satisfying the following properties. Given a $S \in \mathcal{B}_\mathbb{R}$ we would like $\int_X 1_S d\mu = \mu(S)$. Moreover we want continuity, that is if $f_n \to f$ we'd like

$$\lim_{n\to\infty} \int_X f d\mu = \int_X \lim_{n\to\infty} f_n d\mu.$$

To do so we first turn to the notion of measurable functions. Given measurable spaces, $(X, \mathcal{M}), (Y, \mathcal{N})$ we say $f : X \to Y$ is *measurable* if for any $N \in \mathcal{N}$ one has $f^{-1}(N) \in \mathcal{M}$.

{Lem2.1}

Lemma 3.1. *If $\mathcal{N}$ is generated by some collection $\mathcal{E} \subseteq \mathcal{P}(Y)$ then $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$ measurable if and only if for any $E \in \mathcal{E}$ one has $f^{-1}(E) \in \mathcal{M}$.*

Proof. Since unions, and complements are preserved under pre-images, the collection

$$\mathcal{S} = \{B \subseteq Y | f^{-1}(Y) \in \mathcal{M}\},$$

is a σ-algebra. By supposition $\mathcal{E}$ belongs to this σ-algebra and so we have $\mathcal{N} \subseteq \mathcal{S}$. □

{Lem2.2}

Lemma 3.2. *If $f, g : X \to \mathbb{C}$ are $\mathcal{M}$-measurable, then so are $f+g$ and fg.*

Proof. Let $F : X \to \mathbb{C} \times \mathbb{C}$ be given by $F(x) = (f(x), g(x))$. Furthermore let $\phi, \psi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be given by

$$\phi(z, w) = z + w, \quad \psi(z, w) + zw.$$

Evidently F is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}\times\mathbb{C}}$—one easily checks that a map into the product is measurable if and only if its projection onto each term is measurable and sees F is then measurable since $\mathcal{B}_{\mathbb{C}\times C} = \mathcal{B}_\mathbb{C} \otimes \mathcal{B}_\mathbb{C}$. Continuous maps are always measurable on

Borel σ-algebra, and thus ϕ, ψ are also measurable. Thus the result follows since composition of measurable functions are again measurable. □

We can likewise define measurable functions on $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ by insisting the pre-image of any Borel set is measurable along with the pre-image of $\pm\infty$ also being measurable.

{Prop2.3}

Proposition 3.3. *If $\{f_j\}$ is a sequence of $\overline{\mathbb{R}}$ measurable functions with respect to $(X, \mathcal{M})$, then the following functions are also measurable.*

$$g_1(x) = \sup_j f_j(x), \quad g_2(x) = \inf_j f_j(x),$$
$$g_3(x) = \limsup_{j\to\infty} f_j(x), \quad g_4(x) = \liminf_{j\to\infty} f_j(x).$$

Moreover if $f(x) = \lim_{j\to\infty} f_j(x)$ exists for every $x \in X$, then f is measurable.

Proof. Note if $g_1(x) > a$ then by definition there exists some n such that $f_n(x) > a$. Hence

$$g_1^{-1}((a,\infty]) = \bigcup_{j=1}^{\infty} f_j^{-1}((a,\infty]),$$

and so g_1 is measurable as the open rays generate $\mathcal{B}_{\mathbb{R}}$. By an indentical argument using $[-\infty, a)$ one sees g_2 is also measurable. Since $\limsup_{j\to\infty} f_j(x) = \inf_{k=1,2,\ldots} \sup_{n\geq k} f_j(x)$ the previous gives g_3 is measurable and likewise as the limit inferior is defined using sups and infs we get g_4 is also measurable. If $f(x) = \lim_{j\to\infty} f_j(x)$ exists for every $x \in X$, then $g_3 = g_4 = f$ and so f is measurable. □

We now turn our discussion to the so-called *simple functions*. We say $f : X \to \mathbb{R}, \overline{\mathbb{R}}, \mathbb{C}$ is simply if it takes only finitely many values. Fix a measure space $(X, \mathcal{M}, \mu)$, we will let $\mathscr{S}$ denote the collection of all measurrable simple functions on X. Such functions can be written as the fintie sum

$$f = \sum_{i=1}^{k} a_i \mathbf{1}_{A_i},$$

where each $A_i \in \mathcal{M}$. The standard representation of such a function is one where all a_i in the above sume are distinct and non-zero. In this case then the associated $A_1, A_2, \ldots, A_k$ are all pairwise disjoint.

{Lem2.4}

Lemma 3.4. *Let $f : X \to [0,\infty]$ be a measurable non-negative function. Then there exists a collection of simply functions $\varphi_1, \varphi_2, \ldots$ suych that for every $x \in X$*

$$\varphi_1(x) \leq \varphi_2(x) \leq \ldots,$$

and $\varphi_n(x) \to f$ uniformly on any set A such that f is bounded on A.

The book sketches out the details of this construction, so we omit the proof in these notes. Given a non-negative, simple, measurable $\varphi \in \mathscr{S}^+$ we define

$$\int_X \varphi d\mu := \sum_{i=1}^{k} a_i \mu(A_i),$$

where we wriote $f = \sum_{i=1}^{k} a_i \mathbf{1}_{A_i}$. One need check that this definition is well-defined of course, but we leave that as an exercise.

{Lem2.5}

Lemma 3.5. *Let $\varphi, \psi \in \mathscr{S}^+$. Then*

(1) *For any $c \geq 0$, $\int_X c\varphi d\mu = c\int_X \varphi d\mu$.*
(2) *If $\varphi \leq \psi$ then $\int_X \varphi d\mu \leq \int_X \psi d\mu$.*
(3) *$\int_X (\varphi + \psi) d\mu = \int_X \varphi d\mu + \int_X \psi d\mu$.*
(4) *For any $E \in \mathcal{M}$ the assignment*

$$E \mapsto \int_E \varphi d\mu := \int_X \mathbf{1}_E \varphi d\mu,$$

defines a measure.

Proof. We leave the proof of 1,2, and 3 to the reader. Thus we proceed only with the proof of 4. Clearly the empty set maps to 0 under this assignment so we need only show countable additivity. Let $\{A_i\}_{i\in\mathbb{N}}$ be a countable disjoint collection of sets in $\mathcal{M}$ and let $\varphi = \sum_{i=1}^k a_i \mathbf{1}_{E_i}$. Then for $A := \cup_{\mathbb{N}} A_i$

$$\int_A \varphi d\mu = \sum_{j=1}^k a_j \mu(A \cap E_j) = \sum_{j,k} a_j \mu(A_k \cap E_j) = \sum_{k=1}^\infty \int_{A_k} \varphi d\mu.$$

□

3.2. **Integration of Non-negative Functions.** In an effort to define the integral over a broader class of functions we now consider the set

$$L^+ = \{f : X \to [0, \infty] | f \text{ measurable}\}.$$

Guven $f \in L^+(X, \mathcal{M}, \mu)$ we define

$$\int_X f d\mu := \sup\left\{ \int_X \varphi d\mu \middle| \varphi \in \mathscr{S}^+, \varphi \leq f \right\}.$$

By inspection one sees that this definition agrees on the non-negative, measurable, simple functions. We remark here that it immediately follows that items 1 and 2 in Lemma 3.5 for measurable non-negative functions. That is if $f \leq g$ then $\int f \leq \int g$ and for any $c \geq 0$, $c\int f = \int cf$. Additivity of integrals in this case is a bit trickier and thus we postpone the proof until after the proof of the monotone convergence theorem.

{Thm2.6}

Theorem 3.6. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $0 \leq f_1 \leq f_2 \leq \ldots$ be a sequence in L^+. Let $f(x) = \lim_{n\to\infty} f_n(x) = \sup_n\{f_n(x)\}$. Then*

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

Proof. Since $f_n \leq f$ for every n we have $\int f_n \leq \int f$ for every n. From this it follows that

$$\lim_{n\to\infty} \int_X f_n d\mu \leq \int_X f d\mu.$$

For the other direction, we prove that for any $\epsilon > 0$, $\varphi \in \mathscr{S}^+$ with $\varphi \leq f$ one has

$$\lim_{n\to\infty} \int_X f_n d\mu \geq (1 - \epsilon) \int_X \varphi d\mu.$$

This suffices to prove the reverse inequality since taking $\epsilon \to 0$ we'd obtain $\lim_{n\to\infty} \int_X f_n d\mu \geq \int_X \varphi d\mu$ for every $\varphi \leq f$ and so upon taking supremums we'd obtain the desired inequality. To this end, define $E_n := \{x \in X | f_n(x) \geq (1 - \epsilon)\varphi(x)\}$.

By measurability of each E_n we obtain that each E_n is measurable. Moreover since the f_i form an increasing sequence of functions we have $E_1 \subseteq E_2 \subseteq \dots$, and since $\varphi \leq f$ we also have that $\cup_{i\in\mathbb{N}} E_i = X$. Finally, since $\nu(F) = \int_F (1-\epsilon)\varphi d\mu$ defines a measure by Lemma 3.5 we obtain

$$\begin{aligned}\nu(X) = (1-\epsilon)\int_X \varphi d\mu &= \sup \nu(E_n),\\ &= \sup \int_{E_n} (1-\epsilon)\varphi d\mu,\\ &\leq \sup \int_{E_n} f_n d\mu,\\ &\leq \sup \int_X f_n d\mu.\end{aligned}$$

Taking $n \to \infty$ we obtain the desired result. □

{Cor2.7}

Corollary 3.7. *Given $f, g \in L^+$ we have*

$$\int_X (f+g) d\mu = \int_X f d\mu + \int_X g d\mu.$$

Proof. Using Lemma 3.4 there exists increasing sequences of non-negative, measurable, simple functions φ_n, ψ_n such that $\varphi_n \to f, \psi_n \to g$. By the monotone convergence theorem

$$\int_X f d\mu = \lim_{n\to\infty} \int_X \varphi_n d\mu, \quad \int_X g d\mu = \lim_{n\to\infty} \int_X \psi_n d\mu.$$

To conclude we simply observe

$$\int_X f d\mu + \int_X g d\mu = \lim_{n\to\infty}\left(\int_X \varphi_n d\mu + \int_X \psi_n d\mu\right) = \lim_{n\to\infty}\int_X (\varphi_n + \psi_n) d\mu = \int_X (f+g) d\mu.$$

□

{Lem2.8}

Lemma 3.8. *Let $f \in L^+(X, \mathcal{M}, \mu)$, be such that $\int_X f d\mu < \infty$. Then for any $\epsilon > 0$, the set*

$$E_\epsilon := \{x \in X | f(x) \geq \epsilon\},$$

has finite measure and in particular

$$\mu(E_\epsilon) \leq \frac{1}{\epsilon}\int_X f d\mu.$$

Proof. By definition of E_ϵ the function $\epsilon \mathbf{1}_{E_\epsilon} \leq f$ for all $x \in X$. Thus

$$\epsilon\mu(E_\epsilon) = \int_X \epsilon \mathbf{1}_{E_\epsilon} \leq \int_X f d\mu.$$

□

{Cor2.9}

Corollary 3.9. *If $f \in L^+(X, \mathcal{M}, \mu)$ is such that $\int_X f d\mu < \infty$, then $\{x \in X | f(x) > 0\}$ is a σ-finite set. Moreover the set $\{x \in X | f(x) = \infty\}$ has 0 measure. Finally for $f \in L^+$ one has $\int_X f d\mu = 0$ if and only if $f \equiv 0$ μ almost everywhere.*

We omit the proof of this corollary as almost all implications follow immediately from the previous lemma. To conclude this section on integration of positive functions, we discuss the result which is commonly referred to as Fatou's Lemma.

{Lem2.10}

Lemma 3.10. *Let f_n be a sequence of functions in $L^+(X,\mathcal{M},\mu)$. Then*

$$\int_X \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty}\left(\int_X f_n d\mu\right).$$

Proof. Set $g_n(x) = \int_{k\geq n} f_k(x)$. Then for any $j > n$ one has $g_n(x) \leq f_j(x)$ by definition of the infimum. Therefore

$$\int_X g_n d\mu \leq \int_X f_j d\mu.$$

Taking infimums we therefore obtain

$$\int_X g_n d\mu \leq \inf_{j\geq n}\int_X f_j d\mu.$$

Taking $n\to\infty$ on both sides together with the monotone convergence theorem we obtain

$$\int_X \liminf_{n\to\infty} f_n d\mu = \int_X \lim_{n\to\infty} g_n d\mu = \lim_{n\to\infty}\left(\inf_{j\geq n}\int_X f_j d\mu\right) = \liminf_{n\to\infty}\int_X f_n d\mu.$$

□

As the theorem suggests this inequality is, in general, strict. For if we define $f_n(x) = \mathbf{1}_{[0,1/2]}$ for odd n and $f_n(x) = \mathbf{1}_{[1/2,1]}$ for even n we see

$$\liminf_{n\to\infty}\left(\int_X f_n d\mu\right) = 1/2, \quad \int_X \liminf_{n\to\infty} f_n d\mu = 0.$$

3.3. **Integration of General Functions.** Let $(X,\mathcal{M},\mu)$ be a fixed measure space and let $f : X\to\mathbb{R}$ be measurable. Defining

$$f^+ : t = \max(f,0), \quad f^- := \max(-f,0),$$

pointwise we see $f = f^+ - f^-$ and $f^+, f^- \in L^+(X,\mathcal{M},\mu)$. Moreover $|f| = f^+ + f^-$. We then say f is *integrable*, denoted $f\in L^1(X,\mathcal{M},\mu)$ if both

$$\int_X f^+ d\mu < \infty, \quad \int_X f^- d\mu < \infty,$$

and define $\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$..

{Lem2.11}

Lemma 3.11. *$f\in L^1$ if and only if $\int_X |f| d\mu < \infty$. Moreover if $f, g\in L^1$ then $f+g\in L^1$ and*

$$\int_X f d\mu + \int_X g d\mu = \int_X (f+g) d\mu.$$

Proof. The first statment follows from the fact that $|f| = f^+ + f^-$. For the second statement we must get a bit cute. Set $h := f+g$. Then we have

$$h = h^+ - h^- = (f^+ + g^+) - (f^- + g^-).$$

Therefore $f^+ + g^+ + h^- = h^+ + f^- + g^-$. From this we perform the following manipulations.

$$\int f^+ + \int g^+ + \int h^- = \int h^+ + \int f^- + \int g^-,$$
$$\left(\int f^+ + \int g+\right) - \left(\int f^- + \int g^-\right) = \int h^+ - \int h^-,$$
$$\left(\int f^+ - \int f^-\right) + \left(\int g^+ - \int g^-\right) = \int h^+ - \int h^-,$$
$$\int f + \int g = \int h.$$

□

{Prop2.12}

Proposition 3.12. *For any $f \in L^1$ we have*

$$\left|\int_X f d\mu\right| \leq \int_X |f| d\mu.$$

Proof. If f is $\mathbb{R}$-valued, then one need only make the following observation

$$\left|\int_X f d\mu\right| = \left|\int_X f^+ d\mu - \int_X f^-\right| \leq \int_X f^+ d\mu + \int_X f^- d\mu = \int_X |f| d\mu.$$

□

With this in mind, we can now generalize Lemma 3.8 to the context of general functions.

{Lem2.13}

Lemma 3.13. *For $f \in L^1(X, \mathcal{M}, \mu)$ the set set $\{x \in |f(x) \neq 0\}$ is σ finite.*

Proof. The set in question is precisely those points where $|f| > 0$ and so by Lemma 3.8 this set is finite. □

{Lem2.14}

Lemma 3.14. *Let $f, g \in L^1(X, \mathcal{M}, \mu)$. Then the following are equivalent.*

1) $f(x) = g(x)$ for μ almost every $x \in X$.
2) $\int_E f d\mu = \int_E g d\mu$ for every $E \in \mathcal{M}$.
3) $\int_X |f - g| d\mu = 0$.

Proof. The implication *3* ⇒ *1* follows immediately from Corollary 3.9. For *1* ⇒ *2* note that $f \equiv g$ μ almost everywhere gives $\int_X |f - g| = 0$ and so

$$\left|\int_E f d\mu - \int_E g d\mu\right| \leq \int_E |f - g| d\mu \leq \int_X |f - g| d\mu = 0,$$

giving the result. For *2* ⇒ *3* if $\int_X |f - g| d\mu \neq 0$ then $f \neq g$ almost everywhere. Then either $(f - g)^+, (f - g)^-$ must be non-zero on a set of positive meausre, that is we may define $E = \{x \in X | (f - g)^+(x) > 0\}$ with $\mu(E) > 0$.. Then since $(f - g)^- = 0$ on E,

$$\int_E f d\mu - \int_E g d\mu = \int_E (f - g)^+ d\mu > 0,$$

and so by contrapositive we have the result. □

We now define $\mathcal{L}^1(X, \mathcal{M}, \mu)$ to be the quotient of L^1 under the equivalence relation $f \sim g$ if $f(x) = g(x)$ μ almost everywhere. With this new definition we obtain the following.

{Prop2.15}

Proposition 3.15. $\mathcal{L}^1$ *is a vector space and the map*

$$||f||_1 := \int_X |f|d\mu,$$

defines a norm on $\mathcal{L}^1$.

We remark here that the necessity of defining the equivalence relation on L^1 was so that $||\cdot||_1$ satisfied the property that only the 0 function has norm 0. We omit the proof of the proposition as it is a straightforward verification.

{Thm2.16}

Theorem 3.16. *Let* $(X, \mathcal{M}, \mu)$ *be a measure space and* $f_n : X \to \mathbb{R}$ *or* $\mathbb{C}$ *be a sequence in* L^1 *with* $f_n \to f$ *almost everywhere. Assume there exists some* $g \in L^1$ *with* $|f_n| \leq g$ *almost everywhere. Then* $f \in L^1$ *and*

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

Before proceeding with the proof of the above—which is often referred to as the Dominated Convergence Theorem—we present an example of when convergence of integrals fails when we do not assumed that all f_n are dominated by some g. Let $(X, \mathcal{M}, \mu) = [0, 1]$ with the Lebesgue measure. Define $f_n(x) := n\mathbf{1}_{[0,1/n]}$. Then $f_n \to 0$ almost everywhere—the only point of controversy is 0—-and yet $\int_X f_n d\mu = 1$ for all n and hence the integrals do not converge.

Proof of Theorem 3.16. Without loss of generality we may assume that $g \in L^1 \cap L^+$. Since $|f_n| \leq g$ and $\int_X |g|d\mu < \infty$ we obtain that $f \in L^1$ upon taking $n \to \infty$. For every n, $-g \leq f_n \leq g$ and thus $f_n + g \geq 0, g - f_n \geq 0$ giving $f_n + g, g - f_n \in L^+$. By Fatou's Lemma,

$$\begin{aligned}\int_X (f+g)d\mu = \int_X \liminf_{n\to\infty}(f_n+g) &\leq \liminf_{n\to\infty}\left(\int_X f_n + gd\mu\right)\\ &= \liminf_{n\to\infty}\left(\int_X f_n d\mu\right) + \int_X gd\mu.\end{aligned}$$

Since $\int_X gd\mu \leq \infty$, upon subtracting from both sides of the above inequality we obtain

$$\int_X fd\mu \leq \liminf_{n\to\infty}\left(\int_X f_n d\mu\right).$$

Using a similar argument for $g - f_n \in L^+$, we have

$$\begin{aligned}\int_X (g-f)d\mu = \int_X \liminf_{n\to\infty}(g-f_n)d\mu &\leq \liminf_{n\to\infty}\left(\int_X g - f_n d\mu\right)\\ &= \int_X gd\mu - \limsup_{n\to\infty}\left(\int_X f_n d\mu\right),\end{aligned}$$

to obtain $\int_X fd\mu \geq \limsup_{n\to\infty}\left(\int_X f_n d\mu\right)$. □

{Thm2.17}

Theorem 3.17. *Let* $(X, \mathcal{M}, \mu)$ *be a measure space,* $g_1, g_2, \dots : X \to \mathbb{C}$ *be measurable and assume* $\sum_{n=1}^{\infty} ||g_n||_1 < \infty$*. Then* $f(x) := \sum_{n=1}^{\infty} g_n(x)$ *converges absolutely for almost every* $x \in X$.

Proof. Let $F_n(x) = \sum_{k=1}^n |g_k(x)|$. Then $0 \le F_1 \le F_2 \le$ is a monotone sequence of measurable positive functions and so by the monotone convergence theorem, letting $F := \lim_{n\to\infty} F_n(x)$,

$$\int_X F d\mu = \lim_{n\to\infty} \int_X F_n d\mu = \lim_{n\to\infty} \int_X \left(\sum_{k=1}^n |g_k(x)| \right) d\mu = \sum_{k=1}^\infty ||g_k||_1.$$

Therefore $F(x)$ is finite for almost every $x \in X$ and by construction $F(x) = \sum_{k=1}^\infty |g_k(x)|$ and so $\sum_{k=1}^\infty g_k(x)$ converges absolutely for almost every $x \in X$. Thus by the montone convergence theorem for series,

$$\int_X \left(\sum_{k=1}^\infty g_k(x) \right) d\mu = \sum_{k=1}^\infty \left(\int_X g_k(x) d\mu \right).$$

Letting f_n denote the nth partial sum of the series of f, we then obtain

$$\begin{aligned}
\left| \int_X f d\mu - \int_X f_n d\mu \right| &= \left| \int_X (f - f_n) d\mu \right|, \\
&\le \int_X |f - f_n| d\mu, \\
&= \int_X \left| \sum_{k=n+1}^\infty g_k(x) \right| d\mu, \\
&\le \int_X \sum_{k=n+1}^\infty |g_k(x)| d\mu, \\
&= \sum_{k=n+1}^\infty \int_X |g_k(x)| d\mu,
\end{aligned}$$

where the last line follows from the monotone convergence theorem for F. Since the last line tends to 0 as $n \to \infty$ we conclude the result. □

Having already established the normed vector space $\mathcal{L}^1$ we note that the notion of convergence in $\mathcal{L}^1$ is distinct from pointwise converence. To see this we now construct a sequence of L^1 functions who converge to some f in L^1 buit do not converge pointwise. Define $f_n := \mathbf{1}_{[j/2^k,(j+1)/2^k]}$ where $n = 2^k + j$ with $0 \le j < 2^k$. Then $f_n \to 0$ in L^1 as $\int_X f_n d\mu \to 0$ yet one plainly sees that f_n does not converge pointwise to 0.

{Thm2.18}

Theorem 3.18. *$\mathcal{L}^1$ is a complete normed vector space with respect to the norm $||f||_1$.*

Proof. Let f_n be a Cauchy sequence in $\mathcal{L}^1$, that is, for any $\epsilon > 0$, there exists some N such that for all $n > k \ge N$ one has $||f_n - f_k||_1 < \epsilon$. Then we may construct a sequence $N_1 < N_2 < \dots$, such that for all $n > k \ge N_j$ we have

$$||f_n - f_k||_1 \le 2^{-j}.$$

Define a new sequence via $g_j := f_{N_{j+1}} - f_{N_j}$. By construction $||g_j||_1 < 2^{-j}$ and so $\sum_{j=1}^\infty ||g_j||_1 < \infty$. Thus by Theorem 3.17 the function $f(x) = \sum_{k+1}^\infty g_k(x)$ converges for almost every $x \in X$ and $||f - \sum_1^k g_j||_1 \to 0$. Since $\sum_1^k g_j = f_{N_{j+1}}$

we in fact have $||f - f_{N_j}||_1 \to 0$. To conclude, let $\epsilon > 0$, and choose j_0 such that $2^{-j_0} < \epsilon/2$. and $||f - f_{N_j}||_1 < \epsilon/2$ for all $j \geq j_0$. Then for all $n \geq N_{j_0}$ we get

$$||f - f_n||_1 \leq ||f - f_{N_j}|| + ||f_{N_j} - f_n||_1 \leq \epsilon.$$

□

{Thm2.19}

Theorem 3.19. *The collection of integrable simple functions, denoted $\mathcal{S}^1$ is dense in $\mathcal{L}^1$.*

Proof. Let $\epsilon > 0$ and let $f : X \to \mathbb{R}$ be in $\mathcal{L}^1$. Then we have a monotonic sequence of simple functions in L^+ $\varphi_1 \leq \varphi_2 \leq \dots$ such that $\varphi_n \to |f|$. Moreover,

$$|f - \varphi_n| \leq |f| + |\varphi_n| \leq 2|f|,$$

and so by the dominated convergence theorem,

$$\lim_{n\to\infty} \int_X |f - \varphi_n| d\mu = 0.$$

□

We conclude this section with one final statement on the Banach space $\mathcal{L}^1(\mathbb{R}, \mathcal{L}_\mathbb{R}, m)$.

{Thm2.20}

Theorem 3.20. *For every $f \in \mathcal{L}^1(\mathbb{R}, \mathcal{L}_\mathbb{R}, m)$ and any $\epsilon > 0$ there exists a continuous function with compact support ψ such that $||f - \psi||_1 < \epsilon$. That is, the continuous functions with compact support are dense in $\mathcal{L}^1$.*

We omit the proof.

3.4. **Modes of Convergence.** Thus far we have already encountered multiple ways a sequence of real (or complex) valued functions f_n can converge to some f. The ordinary notions of pointwise and uniform convergence from undergraduate analysis still apply although for most purposes we will only require that $f_n \to f$ pointwise or uniformly *almost everywhere*. Having seen that L^1 is a Banach space in the previous section we also have a notion of convergence in the norm on L^1. Namely $f_n \to f$ in L^1 if $||f_n - f||_1 \to 0$ as n tends toward infinity. For a new mode of convergence, we say $f_n \to f$ *in measure* if for any $\epsilon > 0$ the sequence,

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \to 0,$$

as $n \to \infty$.

{Lem2.21}

Lemma 3.21. *If $f_n \to f$ in L^1 then $f_n \to f$ in measure.*

Proof. For any fixed $\epsilon > 0$, we know via Lemma 3.8 that

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \leq \frac{1}{\epsilon} \int_X |f_n - f| d\mu,$$

and since the right-hand quantity tends to 0 as $n \to \infty$ we obtain convergence in measure. □

The converse to Lemma 3.21 is generally false. For a counterexample take $f_n = n\mathbf{1}_{[2^n, 2^n+1/n]}$. Then $f_n \to 0$ pointwise but $\int f_n = 1$ for all n and so f_n does not converge to 0 in L^1. While the uniqueness—up to a nullset—of the limit is clear for pointwise, uniform, and L^1 convergence, the result for convergence in measure requires some justification.

{Lem2.22}

Lemma 3.22. *Suppose $f_n \to g, h$ in measure. Then $g = h$ almost everywhere.*

Proof. Let $\epsilon > 0$ and define the following sets,

$$C_{2\epsilon} := \{x \in X : |g(x) - h(x)| \geq 2\epsilon\},$$
$$A_{n,\epsilon} := \{x \in X : |f_n(x) - g(x)| \geq \epsilon\},$$
$$B_{n,\epsilon} := \{x \in X : |f_n(x) - h(x) \geq \epsilon\}.$$

With these definitions, the triangle inequality then gives that for any $n \in \mathbb{N}$, $C_{2\epsilon} \subseteq A_{n,\epsilon} \cup B_{n,\epsilon}$. By assumption of convergence in measure, $\mu(A_{n,\epsilon}) \to 0$ as $n \to \infty$ and likewise for $\mu(B_{n,\epsilon})$. Hence monotonicity of μ gives $\mu(C_{2\epsilon}) = 0$. Upon noting that

$$\bigcup_{k=1}^{\infty} C_{1/k} = \{x \in X | g(x) \neq h(x)\},$$

we achieve via continuity from below that the set on which g and h differ is indeed a null set. □

We define a sequence of functions f_n to be *Cauchy in measure* if for any $\epsilon > 0, \delta > 0$ there exists an $N \in \mathbb{N}$ such that for all $n, k > N$ one has

$$\mu(\{x \in X : |f_n(x) - f_k(x)| \geq \epsilon\}) < \delta.$$

{Thm2.23}

Theorem 3.23. *Let f_n be a sequence of complex valued, measurable functions which is Cauchy in meausre. Then there exists some measurable $f : X \to \mathbb{C}$ so that $f_n \to f$ in meausre. Moreover, there exists a subsequence f_{n_j} so that $f_{n_j} \to f$ pointwise almost everywhere.*

Proof. Let $(\epsilon_i), (\delta_i)$ be decreasing sequences of positive reals such that

$$\sum_{i=1}^{\infty} \epsilon_i < \infty, \quad \sum_{i=1}^{\infty} \delta_i < \infty.$$

Using the Cauchy in measure condition, we get an increasing sequence $n_1 < n_2 < \dots$, such that for any $l, k > n_j$,

$$\mu(\{x \in X : |f_l(x) - f_k(x)| \geq \epsilon_j\} < \delta_j.$$

Define the set E_j via

$$E_j := \{x \in X : |f_{n_{j+1}}(x) - f_{n_j}(x)| \geq \epsilon_j\}.$$

By construction, for each $j \in \mathbb{N}$ $\mu(E_j) < \delta_j$ and hence $\sum_1^{\infty} \mu(E_j) < \infty$. Setting $E_* = \limsup E_j$, we therefore obtain $\mu(E_*) = 0$. For $x \in X \setminus E_*$, $x \in E_j$ for only finitely many j and hence

$$|f_{n_{j+1}}(x) - f_{n_j}(x)| < \epsilon_j,$$

for any $j > j_0(x)$ where $j_0(x)$ is the maximal $j \in \mathbb{N}$ such that $x \in E_j$. Since $\sum_1^{\infty} \epsilon_j < \infty$, we attain

$$\sum_{j=j_0(x)+1}^{\infty} |f_{n_{j+1}}(x) - f_{n_j}(x)| < \infty,$$

and so $f_{n_j}(x)$ defines a Cauchy sequence in $\mathbb{C}$. Set $f(x) := \lim_{j\to\infty} f_{n_j}(x)$ and define $f(x) = \pi^e$ for $x \in E_*$. Then f is measurable as a pointwise limit of measurable functions. Moreover we claim $f_{n_j} \to f$ in measure. Via the triangle ineqaulity,

$$\left\{x \in X : |f_{n_j}(x) - f(x)| \geq \sum_{i=j}^{\infty} \epsilon_i\right\} \subseteq \bigcup_{i=j}^{\infty} E_i.$$

Since $\mu(\cup_j^\infty E_i) \to 0$ as $j \to \infty$ we obtain convergence in measure. Now let $\epsilon > 0$. By convergence of measure along f_{n_j} there exists some $j' \in \mathbb{N}$ such that for $j \geq j'$,

$$\mu(\{x \in X : |f_{n_j}(x) - f(x)| \geq \epsilon/2\}) < \epsilon/2.$$

Moreover, by the Cauchy criterion, there exists some $N \in \mathbb{N}$ such that for all $l, k N$,

$$\mu(\{x \in X : |f_l(x) - f_k(x)| \geq \epsilon/2\}) < \epsilon/2.$$

Taking $N^* := \max\{n'_j, N\}$, we note that defining for $n > N^*$,

$$\{x \in X : |f_n(x) - f(x)| \geq \epsilon\} \subseteq \{x \in X : |f_n(x) - f_{n'_j}(x)| \geq \epsilon/2\} \cup \{x \in X | f_{n'_j}(x) - f(x)| \geq \epsilon/2\}.$$

By the previous remarks the set on the right hand sides has measure at most ϵ and so $f_n \to f$ in measure. □

We now present what is often referred to as *Egoroff's Theorem*.

{Thm2.24}

Theorem 3.24. *Let $(X, \mathcal{M}, \mu)$ be a finite measure space, f_n be a sequence of measurable functions such that $f_n \to f$ almost everywhere. Then for any $\epsilon > 0$ there exists an $X_\epsilon \subseteq X$ with $\mu(X \setminus X_\epsilon) < \epsilon$ with $f_n \to f$ uniformly on X_ϵ.*

Proof. Let $\epsilon > 0$ and define the set

$$A_{n,k} := \{x \in X : |f_j(x) - f(x)| < 1/k \text{ for all } j \geq n\} = \bigcap_{j=n}^{\infty} \{x \in X : |f_j(x) - f(x)| < 1/k\}.$$

We note that $A_{1,k} \subseteq A_{2,k} \subseteq \dots$ for all k. Moreover, for any fixed k, $\mu(X \setminus \cup_1^\infty A_{n,k}) = 0$ since $f_n \to f$ almost everywhere. Set $B_{n,k} = X \setminus A_{n,k}$ and note that $B_{1,k} \supseteq B_{2,k} \supseteq \dots$, and $\mu(\lim_{n\to\infty} B_{n,k}) = 0$ via continuity from above—here we are using the fact that $\mu(X) < \infty$. Choose $n_1 < n_2 < \dots$ so that for ech j,

$$\mu(B_{n_j,k}) < \epsilon 2^{-j}.$$

Set $B := \cup_1^\infty B_{n_j,k}$ and let $X_\epsilon = X \setminus B$. We notice

$$\mu(B) \leq \sum_{j=1}^{\infty} \mu(B_{n_j,k}) < \epsilon.$$

Moreover, for any $x \in X_\epsilon$, then $x \in A_{n,k}$ for any $n \geq n_1$ and hence $|f_n(x) - f(x)| < 1/k$ for all $n \geq n_1$ giving uniform convergence. □

{Lem2.25}

Lemma 3.25. *Suppose $f_n \to f$ almost everywhere and assume there exists some $g \in L^1$ with $|f_n| \leq g$ for all n. Then $f \in L^1$ and $f_n \to f$ in L^1.*

Proof. Since each $|f_n| \leq g$ upon taking limits we achieve $|f| \leq g$ almost everywhere and hence $f \in L^1$. Moreover for any n,

$$|f_n - f| \leq |f_n| + |f| \leq 2|g|,$$

almost everywhere. Hence by the dominated convergence theorem, $\int |f_n - f| \to 0$, giving convergence in L^1. □

3.5. **Applications of the Dominated Convergence Theorem.** In this section we provide two useful applications of the dominated convergence, the second of which relates the Lebesgue Integral to the Riemann Integral.

{Thm2.26}

Theorem 3.26. *Suppose* $f : X \times [a, b] \to \mathbb{C}$ *with* $-\infty < a < b < \infty$ *is such that* $f(\cdot, t) : X \to \mathbb{C}$ *is integrable for all* $t \in [a, b]$. *Set* $F(t) = \int_X f(x, t) d\mu(x)$. *Then*

a) *If there exists* $g \in L^1(\mu)$ *such that* $|f(x, t)| \leq g(x)$ *for all* $x \in X, t \in [a, b]$, *and if* $\lim_{t \to t_0} f(x, t) = f(x, t_0)$ *for all* $x \in X$, *then* $\lim_{t \to t_0} F(t) = F(t_0)$. *In particular, if* $f(x, \cdot) : [a, b] \to \mathbb{C}$ *is continuous for every* $x \in X$, *then* F *is continuous.*

b) *If* $\partial f / \partial t$ *exists and there exists* $g \in L^1(\mu)$ *with*

$$\left| \frac{\partial f}{\partial t}(x, t) \right| \leq g(x),$$

for all $x \in X, t \in [a, b]$, *then* F *is differentiable and*

$$F'(t) = \int_X \frac{\partial f}{\partial t}(x, t) d\mu(x).$$

Proof. a): Let $(t_n)_{n \geq 1} \subseteq [a, b]$ be such that $\lim_{n \to \infty} t_n = t_0$. Then by assumption for every $x \in X$ $\lim_{n \to \infty} f(x, t_n) = f(x, t_0)$ and $|f(x, t_n)| \leq g(x)$ for all $x \in X, n \in \mathbb{N}$. Hence by the Dominated Convergence theorem,

$$\lim_{t \to t_0} F(t_0) = \lim_{n \to \infty} F(t_n) = \lim_{n \to \infty} \int_X f(x, t_n) d\mu(x) = \int_X f(x, t_0) d\mu(x).$$

b): We remark first that $\frac{\partial f}{\partial t}(x, t)$ is mesurable. To see this let $t_0 \in (a, b), t_n$ be a sequence converging to t_0. Then

$$\frac{\partial f}{\partial t}(x, t_0) = \lim_{n \to \infty} \frac{f(x, t_n) - f(x, t_0)}{t_n - t_0}.$$

With the rightmost quantity above being a limit of measurable functions, we obtain $\partial f / \partial t$ is indeed measurable. For brevity, define

$$h_n(x) := \frac{f(x, t_n) - f(x, t_0)}{t_n - t_0}.$$

Then by the Mean-Value Theorem,

$$|h_n(x)| \leq \sup_{t \in [a, b]} \left| \frac{\partial f}{\partial t}(x, t) \right| \leq g(x).$$

Thus via the dominated convergence theorem,

$$\int_X \frac{\partial f}{\partial t}(x, t_0) d\mu(x) = \lim_{n \to \infty} \int_X \frac{f(x, t_n) - f(x, t_0)}{t_n - t_0} = \lim_{n \to \infty} \frac{F(t_n) - F(t_0)}{t_n - t_0} = F'(t_0).$$

□

{Thm2.27}

Theorem 3.27. *Let* f *be a bounded function on* $[a, b] \subseteq \mathbb{R}$. *If* f *is Riemann Integrable, then* f *is Lebesgue measurable—and hence integrable as it is bounded on a bounded interval—and*

$$\int_a^b f(x) dx = \int_{[a, b]} f dm.$$

Proof. For any partition P of $[a,b]$, let

$$G_P = \sum_{i=1}^n M_i \mathbf{1}_{(t_{i-1},t_i]}, \quad g_P = \sum_{i=1}^n m_i \mathbf{1}_{(t_{i-1},t_i]},$$

where $M_i = \sup_{(t_{i-1},t_i]} f(x), m_i = \inf_{(t_{i-1},t_i]} f(x)$. Letting S_P, s_P denote the upper and lower sums for P respectively, we note that

$$S_P = \int G_P dm, \quad s_P = \int g_P dm.$$

There exists some sequence of partitions P_k with the width of P_k tending to 0 and with S_{P_k}, s_{P_k} both tending towards $\int_a^b f dx$ as $k \to \infty$. Let $G := \lim_{k\to\infty} G_{P_k}, g := \lim_{k\to\infty} g_{P_k}$. Then $g \leq f \leq G$ and, since all of these functions are bounded on a bounded interval, by the dominated convergence theorem

$$\int G dm = \int g dm = \int_a^b f(x) dx.$$

Thus $\int_{[a,b]} (G-g) dm = 0$ and so $G = g$ almost everywhere giving $G = f$ almost everywhere. G is measurable as the limit of a sequence of measurable simple functions, and hence f is measurable—since m is complete. Moreover,

$$\int_{[a,b]} f dm = \int_{[a,b]} G dm = \int_a^b f(x) dx.$$

□

3.6. **Product Measures.** Given two measure spaces $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ the goal of this section is turned the product $X \times Y$ into a measure space which in some sense is built out of $\mathcal{M}$ and $\mathcal{N}$. We define the σ-algebra $\mathcal{M} \otimes \mathcal{N}$ to be the smallest σ algebra containing all sets of the form $A \times B$ where $A \in \mathcal{M}$ and $B \in \mathcal{N}$. Alternatively, this is the smallest σ-algebra such that the projections $X \times Y \to X$ and $X \times Y \to Y$ are measurable. Define the following algebra,

$$\mathcal{A} = \Big\{ \bigsqcup_{i=1}^n A_i \times B_i | n \in \mathbb{N}, A_i \in \mathcal{M}, B_i \in \mathcal{N} \Big\}.$$

{Lem2.28}

Lemma 3.28. *For $E = \bigsqcup_1^n A_i \times B_i \in A$, the function*

$$\pi(E) = \sum_{i=1}^n \mu(A_i)\nu(B_i),$$

defines a premeasure on $\mathcal{A}$.

Proof. □

Now using all of the machinery that was established in Chapter 1, π extends to an outer measure on $X \times Y$ which restricts to a measure on a sigma algebra which necessarily contains $\mathcal{M} \otimes \mathcal{N}$. Moreover on $\mathcal{A}$ this measure takes the values of π.

{Lem2.29}

Lemma 3.29. *Let $E \in \mathcal{M} \otimes \mathcal{N}$. For $x \in X$, set*

$$E_x := \{y \in Y | (x,y) \in E\},$$

and likewise for $y \in Y$ define

$$E_y := \{x \in X | (x,y) \in E\}.$$

Then for any $x \in X, y \in Y$, $E_x \in \mathcal{N}, E_y \in \mathcal{M}$.

Proof. Let

$$W := \{E \subseteq X \times Y | E_x \in \mathcal{N} \text{ for any } x \in X, E_y \in \mathcal{M} \text{ for any } y \in Y\}.$$

Then one easilly sees W forms a σ-algebra. Moreover it contains all sets of the form $A \times B$ for $A \in \mathcal{M}, B \in \mathcal{N}$ and thus contains $\mathcal{M} \otimes \mathcal{N}$. □

{Lem2.30}

Lemma 3.30. *If $f : X \times Y \to \mathbb{C}$ is $\mathcal{M} \otimes \mathcal{N}$ measurable then for any $x \in X$, the function $f_x : Y \to \mathbb{C}$ defined by $f_x(y) = f(x,y)$ is $\mathcal{N}$-meausrable and the function $f_y : X \to \mathbb{C}$ given by $f_y(x) = f(x,y)$ is $\mathcal{M}$-meausrable.*

Proof. Since this a symmetric statement, we show only that f_x is $\mathcal{N}$-meausrable. Let $B \in \mathcal{N}$. Then

$$f_x^{-1}(B) = f^{-1}(B)_x \in \mathcal{N},$$

by Lemma 3.29. □

We now state but do not prove the somewhat tedious monotone class lemma in order to aid in the proof of the Fubini-Tonelli Theorem. For some ambient space X, $\mathscr{C} \subseteq \mathscr{P}(X)$ is said to be *monotone class* if $\mathscr{C}$ is closed under countable increasing unions and countable decreasing intersections. Fixing notation for $\mathscr{E} \subseteq \mathscr{P}(X)$ we let $\mathscr{C}(\mathscr{E})$ denote the smallest monotone class containing $\mathscr{E}$.

{Lem2.31}

Lemma 3.31. *Let $\mathcal{A}$ be an algebra on X. Then $\mathscr{C}(\mathcal{A})$ is the σ-algebra generated by $\mathcal{A}$.*

{Thm2.32}

Theorem 3.32. *Let $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ be σ-finite measure spaces. Then for every $E \in \mathcal{M} \otimes \mathcal{N}$ the maps $X \to [0, \infty]$ $Y \to [0, \infty]$ given by*

$$x \mapsto \nu(E_x), \quad y \mapsto \mu(E_y),$$

define measurable functions and

$$(\mu \times \nu)(E) = \int_X \nu(E_x) d\mu = \int_Y \mu(E_y) d\nu.$$

Proof. Assume that μ, ν are both finite. Define the following collection,

$$\mathscr{C} = \left\{ E \subseteq X \times Y \middle|^{x \mapsto \nu(E_x), y \mapsto \mu(E_y) \text{ are both measurable and}}_{(\mu \times \nu)(E) = \int_X \nu(E_x) d\mu = \int_Y \mu(E_y) d\nu} \right\}.$$

We claim that $\mathscr{C}$ is a montone class which contains the algebra $\mathcal{A}$. The containment $\mathcal{A} \subseteq \mathscr{C}$ is plainly seen to be true and so we only need check that $\mathscr{C}$ is a montone class. Suppose that $\{E_n\}_{n \in \mathbb{N}}$ define an increasing sequence of sets belonging to $\mathscr{C}$ and set $E = \bigcup_1^\infty E_n$. Define the functions $f_n : Y \to [0, \infty]$ via $f(y) = \mu((E_n)_y)$. By hyptohesis each f_n is measurable and since the E_n form an increasing sequence, we have that $f_1 \leq f_2 \leq \dots$ and $f_n \to \mu(E_y)$ are $n \to \infty$. The monotone convergence theorem then implies

$$\int_Y \mu(E_y) d\nu(y) = \lim_{n \to \infty} \int_Y \mu((E_n)_y) d\nu(y) = \lim_{n \to \infty} (\mu \times \nu)(E_n) = (\mu \times \nu)(E),$$

where the last equality follows from continuity from below. By assuming finitness of X, a similar argument using continuity from above and the dominated convergence theorem implies that $\mathscr{C}$ is closed under countable decreasing sequences. Following the standard procedure to pass from finite measure spaces to σ-finite measure spaces, the proof is then complete by the montone class lemma. □

We now present what is often referred to as the *Fubini-Tonelli Theorem.* The first statement therein being due to Tonelli and second to Fubini.

{Thm2.33}

Theorem 3.33. *Suppose $(X,\mathcal{M},\mu),(Y,\mathcal{N},\nu)$ are σ-finite measure spaces.*

a) Suppose $f\in L^+(\mu\times\nu)$. Then $f_x\in L^+(\nu)$ and $f_y\in L^+(\mu)$ and the assignments,

$$x\mapsto\int_Y f_x d\nu(y),\quad y\mapsto\int_X f_y d\mu,$$

are both measurable and

$$\int_{X\times Y} fd(\mu\times\nu)=\int_X\left[\int_Y f_x d\nu(y)\right]d\mu(x)=\int_Y\left[\int_X f_y d\mu(x)\right]d\nu(y).$$

b) Suppose $f\in L^1(\mu\times\nu)$. The for μ almost every $x\in X$, $f_x\in L^1(\nu)$ and for ν almost every $y\in Y$ $f_y\in L^1(\mu)$. Moreover the almost everywhere assignments,

$$x\mapsto\int_Y f_x d\nu(y),\quad y\mapsto\int_X f_y d\mu,$$

belong to $L^1(\mu)$ and $L^1(\nu)$ respectively. Finally,

$$\int_{X\times Y} fd(\mu\times\nu)=\int_X\left[\int_Y f_x d\nu(y)\right]d\mu(x)=\int_Y\left[\int_X f_y d\mu(x)\right]d\nu(y).$$

Proof. For a, we note that this is already proven in the case where f is a characteristic function via Theorem 3.32. By linearity this means this holds true for all measurable simple functions. Let $g(x)=\int_Y f_x d\nu(y), h(y)=\int_X f_y d\mu$. There exists a sequence of measurable simple functions f_n increasing pointwise to f and by the montone convergence theorem, the corresponding g_n and h_n increase pointwise to g and h respectively. Using the montone convergence theorem again we then obtain

$$\int_X gd\mu=\lim_{n\to\infty}\int_X g_n d\mu=\lim_{n\to\infty}\int_{X\times Y} f_n d(\mu\times\nu)=\int_{X\times Y} fd(\mu\times\nu),$$
$$\int_Y hd\nu=\lim_{n\to\infty}\int_Y h_n d\nu=\lim_{n\to\infty}\int_{X\times Y} f_n d(\mu\times\nu)=\int_{X\times Y} fd(\mu\times\nu),$$

where the middle equality each times using the fact that the f_n are simple. This proves Tonelli's theorem. Moreover this shows that if $\int_{X\times Y} fd(\mu\times\nu)<\infty$ then both $g,h<\infty$ almost everywhere. Thus $g,h\in L^1$ and so applying this to the positive or negative parts of f we conclude Fubini's theorem as well. □

3.7. **The Lebesgue Meausre on $\mathbb{R}^n$.** We now investigate the properties of the Lebesgue measure on $\mathbb{R}^n$ viewed as the product measure $(\mathcal{L},m)^{\otimes n}$. First we establish the extension of the regularity properties of the Lebesgue measure.

{Prop2.34}

Proposition 3.34. *The measure space $(\mathbb{R}^n,\mathcal{L}_{\mathbb{R}^n},m_n)$ has the following properties.*

i. For every $E\in\mathcal{L}_{\mathbb{R}^n}$,

$$\begin{aligned}m_n(E)&=\inf\{m_n(U)|U \text{ open and } E\subseteq U\},\\&=\sup\{m_n(K)|K \text{ compact and } K\subseteq E\}.\end{aligned}$$

ii. For any compact $K\subseteq\mathbb{R}^n$, $m_n(K)<\infty$ and for any open $U\subseteq\mathbb{R}^n$, $m_n(U)>0$.

iii. *For any finite $E \in \mathcal{L}_{\mathbb{R}^n}$ and any $\epsilon > 0$, there exsists a finite disjoint union of rectangles $R_1, \ldots, R_n$ where*

$$R_i = [a_i^1, b_i^1] \times \cdots \times [a_i^n, b_i^n],$$

and $m(E \Delta \bigsqcup R_i) < \epsilon$.

iv. *The continuous functions of compact support are dense in $L^1(m_n)$.*

v. *m_n is translation invariant. That is, for any $\tau_a : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\tau_a(x) = x + a$ and for any $E \in \mathcal{L}_{\mathbb{R}^n}$ one has $m_n(E) = m_n(\tau_a(E))$. Moreover if $f \in L^1(m_n)$ then*

$$\int_{\mathbb{R}^n} f(x+a) dm_n(x) = \int_{\mathbb{R}^n} f(x) dm_n(x).$$

We omit the proof of these statements as they are easy consequences of the result on $\mathbb{R}$.

4. Signed Measures and Dirfferentiation

4.1. **Signed Measures.** Let X be any space, $\mathcal{M}$ be a σ-algebra on X. A *signed measure* on $\mathcal{M}$ is a map $\nu : \mathcal{M} \to \mathbb{R}, [\infty, \infty), (-\infty, \infty]$ such that $\nu(\varnothing) = 0$ and for any collection of disjoint $E_i \in \mathcal{M}$,

$$\nu\Big(\bigsqcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \nu(E_i).$$

We remark that this definition implictly assumes absolute convergence of the above sum since it is well known that the terms in a conditionally convergent series can be rearranged so that the infinite sum is any value. As a first example of a signed measure, let μ_1, μ_2 be arbitrary measures on $(X, \mathcal{M})$ and set $\nu = \mu_1 - \mu_2$. Given a signed measure ν on $(X, \mathcal{M})$, we say $E \in \mathcal{M}$ is totally positive (or ν-positive) if all measurable subsets of E have non-negative ν-measure. Likewise E is totally negative if every subset of E has non-positive ν-meausre and E is totally null if every subset of E has zero ν-measure.

{Thm3.1}

Theorem 4.1 (Hahn's Theorem)**.** *Let ν be a signed measure on $(X, \mathcal{M})$. Then there exists a measurable partition $X = P \cup N$ where $P, N \in \mathcal{M}$ so that P is totally positive and N is totally negative. Moreover such a partition is unique up to ν-null sets.*

The proof of Hahn's Theorem will require a bit of work so we first prove a basic lemma.

{Lem3.2}

Lemma 4.2. *If A is a totally positive set and $A_1 \subseteq A$ then A_1 is totally positive. Moreover if $A_1, A_2, \ldots$ is a sequence of totally positive sets then $A = \bigcup_1^\infty A_i$ is totally positive.*

Proof. The first item is immediately obvious from the definitions. For the second item, we may assume the sequence of A_i are disjoint since otherwise we would simply set each $A_i = A_i \setminus (A_1 \cup \cdots \cup A_{i-1})$. For any $B \subseteq A$ we have

$$B = \bigsqcup_{i=1}^{\infty} A_i \cap B.$$

Hence

$$\nu(B) = \sum_{i=1}^{\infty} \nu(A_i \cap B) \geq 0,$$

since each A_i is totally positive. □

We also remark that signed measures remain continuous from above and below with the proofs of these facts being very similar to the proof for ordinary measures. One must take some care in these proofs however, as signed measure are not generally montonic.

Proof of Theorem 4.1. Without loss of generality we assume that ν does not take the value ∞. Set $\alpha = \sup\{\nu(A) | A \in \mathcal{M} \text{ totally positive }\}$. We claim that $\alpha < \infty$. For if this were the case we'd have the existence of $A_1, A_2, \dots$ with $\nu(A_i) \to \infty$. Continuity from below would then implies that

$$\nu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \infty,$$

contradicting the fact that ν did not take the value ∞. Hence $\alpha < \infty$ and moreover this proof shows that there exists $P \in \mathcal{M}$ such that $\nu(P) = \alpha$.

We now claim that $N := P^c$ is ν-totally negative. To prove this we assume the contrary, reaching for a contradiction. We remark first that N cannot contain any totally positive sets $E \subseteq N$ which are not totally null. This is because if $E \subseteq N$ satisfies $\nu(E) > 0$, then $\nu(E \cup P) = \nu(E) + \nu(P) > \alpha$. Hence E cannot belong be totally positive. Moreover if $E \in \mathcal{M}, E \subseteq N$ satisfies $\nu(E) > 0$ then there must exist some $E' \subseteq E$ such that $\nu(E') > \nu(E)$. To see this note that since E is not totally positive there exists $F \subseteq E$ with $\nu(F) < 0$. Setting $E' = E \setminus F$ we get

$$\nu(E) = \nu(E') + \nu(F) < \nu(E').$$

Having assumed that N is not totally negative, we construct the following sequence of sets. We define n_1 to be the smallest integer such that there exists some $E \subseteq N$ with $\nu(E) > n_i^{-1}$ and set A_1 to be such a set. By the previous remarks, we may then define inductively n_i to be the smallest integer such that there exists $F \subseteq E_{i-1}$ with

$$\nu(F) > \nu(A_{i-1}) + n_i^{-1},$$

and A_i to be such a set. To conclude, we set $A = \bigcap_1^{\infty} A_i$. By assumption $\nu(A) < \infty$ and by continuity from above,

$$\infty > \nu(A) = \lim_{i \to \infty} \nu(A_i) > \sum_{i=1}^{\infty} \frac{1}{n_i}.$$

In particular this implies $n_i \to \infty$ as $i \to \infty$. But again by the previous remarks there exists some $B \subseteq A$ with $\nu(B) > \nu(A) + n^{-1}$ for some $n \in \mathbb{N}$. For sufficiently large j, $n < n_j$ and $B \subseteq A_{j-1}$ which contradicts the construction of the A_j.

Uniqueness of the theorem is an easy exercise. □

Given ordinary measures ν and μ on $(X, \mathcal{M})$ we say that ν and μ are *mutually singular* and write $\nu \perp \mu$ if there exists $X = E \sqcup F$ with $E, F \in \mathcal{M}$ such that

$\nu(F) = 0$ and $\mu(E) = 0$. For example letting $\{x_n | n \in \mathbb{N}\}$ be any countable subset of $\mathbb{R}$ and defining ν via

$$\nu = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{x_n}.$$

Then $m \perp \nu$.

{Thm3.3}

Theorem 4.3 (Jordan Decomposition)**.** *Let ν be a signed measure on $(X, \mathcal{M})$. Then there exists unique measures ν^+ and ν^- on $(X, \mathcal{M})$ so that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.*

Proof. Set $\nu^+ = \nu|_P, -\nu^-|_N$, with P and N as in Hahn's decomposition. □

We also define the *total variation measure* as $|\nu| = \nu^+ + \nu^-$. We define integration with respect a signed measure ν on $(X, \mathcal{M})$ by setting $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$ and

$$\int f d\nu = \int f d\nu^+ - \int f d\nu^-.$$

4.2. **Radon-Nikdoym Derivatives.** Given a signed measure ν on $(X, \mathcal{M})$ and μ some ordinary measure, we say that ν is *absolutely continuous* with respect to μ, written $\nu \ll \mu$ if for any $E \in \mathcal{M}$ such that $\mu(E) = 0$, we have $\nu(E) = 0$. For example let $f \in L^+(X, \mathcal{M}, \mu)$ and define ν on $(X, \mathcal{M}$ via $\nu(E) = \int_E f d\mu$. Then clearly $\nu \ll \mu$. The main theorem of this section will show that in the case of σ-finite measure spaces, these are the only examples of absolutely continuous measures.

{Lem3.4}

Lemma 4.4. *Suppose $\nu \ll \mu$ and $\nu(X) < \infty$. Then for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $E \in \mathcal{M}$, if $\mu(E) < \delta$ then $|\nu(E)| < \epsilon$.*

Proof. Assume by way of contradiction that there exists some $\epsilon_0 > 0$ such that for all $n \in \mathbb{N}$ there exists some $E_n \in \mathcal{M}$ with $\mu(E_n) < 2^{-n}$ but $\nu(E_n) \geq \epsilon_0$. Since $\sum_{n=1}^{\infty} \mu(E_n) < \infty$ we have that $\mu(\limsup E_n) = 0$. On the other hand, $\nu(E) \geq \epsilon_0 > 0$ contradicting absolute continuity. □

An important corollary of this lemma is the following.

{Cor3.5}

Corollary 4.5. *Given a measure space $(X, \mathcal{M}, \mu)$ and an $f \in L^1(\mu)$, for any $\epsilon > 0$, there exists a δ such that for any $E \in \mathcal{M}$ with $\mu(E) < \delta$,*

$$\left| \int_E f d\mu \right| < \epsilon.$$

This follows from the fact that the assignment $E \mapsto \int_E f d\mu$ defines a measure on $\mathcal{M}$ which is absolutely continuous with respect to μ.

{Thm3.6}

Theorem 4.6 (Lebesgue-Radon-Nikodym Theorem)**.** *Let ν be a σ-finite measure on $(X, \mathcal{M})$, μ a σ-finite measure. There exists a unique decomposition $\nu = \rho + \lambda$ where $\rho \perp \lambda$ and $\lambda \ll \mu$. Moreover these is a unique $f \in L^1(X, \mathcal{M}, \mu)$ such that*

$$\lambda(E) = \int_E f d\mu,$$

for any $E \in \mathcal{M}$.

Before proceeding with the proof we fix notation and terminology. The function f in the statement of the theorem is sometimes wirtten as

$$f(x) = \frac{d\lambda}{d\mu}(x),$$

or $d\lambda = f d\mu$. f is commonly referred to as the Radon-Nikodym derivative of λ with respect to μ.

Proof of Theorem 4.6. We first restrict to the case where μ and ν are both finite, positive measures. Consisder,

$$\mathcal{F} = \Big\{ f \in L^+(\mu) | \text{ for every } E \in \mathcal{M}, \nu(E) \geq \int_E f d\mu \Big\}.$$

We first observe that if $f_1, f_2 \in \mathcal{F}$ then $\max(f_1, f_2) \in \mathcal{F}$. Indeed writing $X = X_1 \sqcup X_2$ where $X_1 = \{x | f_1(x) \geq f_2(x)\}$ and $X_2 = X_1^c$, we see for any $E \in \mathcal{M}$, $\nu(E) = \nu(E \cap X_1) + \nu(E \cap X_2)$. Therefore

$$\begin{aligned} \nu(E) &= \nu(E \cap X_1) + \nu(E \cap X_2), \\ &\geq \int_{E \cap X_1} f_1 d\mu + \int_{E \cap X_2} f_2 d\mu, \\ &= \int_E \max(f_1, f_2) d\mu. \end{aligned}$$

Moreover if $g_1 \leq g_2 \leq \dots$, with each $g_i \in \mathcal{F}$ then $\lim g_i \in F$. Indeed since for each n, $\nu(E) \geq \int_E g_n d\mu$, using the Monotone Convergence Theorem,

$$\nu(E) \geq \lim_{n \to \infty} \int_E g_n d\mu = \int_E \lim g_n d\mu.$$

Combining these two observations we see for any collection $f_1, f_2, \dots \in \mathcal{F}$ one has $\sup\{f_n | n \in \mathbb{N}\} \in \mathcal{F}$.

Let $t := \sup \Big\{ \int_X f d\mu | f \in \mathcal{F} \Big\}$. We note that $t < \infty$ since $t \leq \nu(X) < \infty$. Let $f_n \in \mathcal{F}$ be such that $\int_X f_n \to t$ and set $f := \sup\{f_n | n \in \mathbb{N}\} \in \mathcal{F}$. We claim that f is the desired Radon-Nikodym derivative. In order to show this we must appropriately define λ and ρ. To this end, define $\lambda(E) := \int_E f d\mu$ for each $E \in \mathcal{M}$. Then by construction for any $E \in \mathcal{M}, \lambda(E) \leq \nu(E)$. Thus we define $\rho := \nu - \lambda$.

By construction all of the result has been shown except for the statement that $\rho \perp \lambda$. By the Dominated Convergence Theorem $t = \int_X f d\mu$. For each $n \in \mathbb{N}$, define the signed measure $\theta_n := \rho - \frac{1}{n}\mu$, and let $P_n \sqcup N_n$ be the Hahn decomposition for θ_n. For each $\rho \subseteq P_n$, we have $\rho(E) \geq n^{-1}\mu(E)$. Hence,

$$\rho(E) = \nu(E) - \int_E f d\mu \geq \frac{1}{n}\mu(E),$$

giving $\nu(E) \geq \int_E (f + 1/n) d\mu$. Thus $f + 1/n\mathbf{1}_{P_n} \in \mathcal{F}$ for each n. But by definition of t we have

$$\int f + \frac{1}{n} P_n d\mu = t + \frac{1}{n}\mathbf{1}_1 P_n d\mu = t + \frac{1}{n}\mu(P_n) \leq t.$$

This implies $\mu(P_n) = 0$ for each n and thus setting $P := \bigcup_1^\infty P_n$ we have $\mu(P) = 0$. To conclude singularity it remains to show $\rho(X \setminus P) = 0$. $\square$

{Prop3.7}

Proposition 4.7. *Let ν be a σ-finite signed measure on $(X, \mathcal{M})$ with μ, λ positive measures such that $\nu \ll \mu$ and $\mu \ll \lambda$. Then,*

a) if $g \in L^1(\nu)$, then $g(d\nu/d\mu) \in L^1(\mu)$ and

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu.$$

b) Moreover $\nu \ll \lambda$, and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda},$$

λ almost everywhere.

Proof. The first statement is true for characteristic functions and hence true for simple functions by linearity. Using the monotone convergence theorem it is therefore true to positive integrable functions and by linearity thus true for all $g \in L^1(\nu)$.

For the second statement we let $E \in \mathcal{M}$ and set $g = \mathbf{1}_E(d\nu/d\mu)$ and use the previous to note,

$$\int_E \frac{d\nu}{d\lambda} d\lambda = \nu(E) = \int_E \frac{d\nu}{d\mu} d\mu = \int g d\mu = \int g \frac{d\mu}{d\lambda} d\lambda = \int_E \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} d\lambda.$$

giving the claim. □

4.3. **Lebesgue Differentiation.** For this section we restrict to the measure space $(\mathbb{R}^n, \mathcal{L}, m)$. We define $f : \mathbb{R}^n \to \mathbb{R}$ to be locally integral, written $f \in L^1_{\text{loc}}$ if for every compact $K \subseteq R^n$, $\mathbf{1}_K f \in L^1(\mathbb{R}^n, \mathcal{L}, m)$. For motivation of this definition we note there are two ways a function may not have finite integral. Namely f could tend towards infinity to quickly in some neighborhood of a point $x \in \mathbb{R}^n$, or f could fail to not settle down fast enuogh at infinity. L^1_{loc} allows for functions of the second type. We will build up many similar statements of Lebesgue differentiation in this section, the first of which being the following.

{Thm3.8}

Theorem 4.8 (Lebesgue Differentiation 1)**.** *If $f \in L^1_{loc}$, then for m almost every $x \in \mathbb{R}^n$;*

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f dm = f(x).$$

In particular, for $E \in \mathcal{L}$ one has

$$\lim_{r \to 0} \frac{m(E \cap B(x,r))}{m(B(x,r))} = \mathbf{1}_E(x),$$

for almost every $x \in \mathbb{R}^n$.

To prove this statement we establish a few supporting facts, the first of which being Vitali's Covering Lemma.

{Lem3.9}

Lemma 4.9 (Vitali's Covering Lemma)**.** *There exists a universal constant C_n such that for every bounded $A \in \mathcal{L}$ and any cover $A \subseteq \bigcup_\alpha B(x_\alpha, r_\alpha)$, there exists a finite subcover, indexed by $\alpha_1, \ldots, \alpha_k$, so that the $B(x_{\alpha_i}, r_{\alpha_i})$ are disjoint and,*

$$m\left(\bigsqcup_{i=1}^{k} B(x_{\alpha_i}, r_{\alpha_i})\right) \geq \frac{1}{C_n} m(A).$$

Proof. Let $K \subseteq A$ be compact with $m(K) > m(A) - \epsilon$ for some fixed $\epsilon > 0$. There exists a finite subcover of the balls which cover K, by compactness. We let α_1 correspond to the ball of maximum radius in this finite cover and set $B_1 = B(x_{\alpha_1}, r_{\alpha_1})$. We then let α_2 correspond the ball of maximum radius in the cover of K which does not intersect B_1. We repeat this process inductively to find out $\alpha_1, \dots, \alpha_k$ and for brevity write $B_i = B(x_{\alpha_i}, r_{\alpha_i})$. Observe that

$$\bigcup_{i=1}^{k} B(x_{\alpha_i}, 3r_{\alpha_i}) \supseteq K,$$

since any ball not a part of this union has non-trivial intersection with a ball of necessarily larger radius. Then,

$$\begin{aligned} m(A) - \epsilon \leq m(K) \leq m\Big(\bigcup_{i=1}^{k} B(x_{\alpha_i}, 3r_{\alpha_i})\Big) &\leq \sum_{i=1}^{k} m(B(x_{\alpha_i}, 3r_{\alpha_i}), \\ &\leq 3^n m\Big(\bigsqcup_{i=1}^{k} B(x_{\alpha_i}, r_{\alpha_i})\Big). \end{aligned}$$

Taking $\epsilon \to 0$ attains the result. □

Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and define for $r > 0$,

$$A_r f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f dm.$$

We see that $A_r f$ is the linear operator which averages the values around some point $x \in \mathbb{R}^n$.

{Lem3.10}

Lemma 4.10. *$A_r f(x)$ is continuous in both r and x.*

Proof. For continuity in x simply note,

$$|A_r f(x) - A_r f(y)| \leq \frac{1}{m(B(x,r))} \int_{B(x,r)\Delta B(y,r)} |f| dm,$$

and as $x \to y$, the symmetric difference in the integral approaches a set of measure 0. Continuity in r follows similarly as $B(x,r)\Delta B(x,r') = \{y \in \mathbb{R}^n \,|r' < |y-x| \leq r\}$. □

We define the *Hardy-Littlewood Maximal Function* via $Hf(x) = \sup_{r>0} A_r f(x)$. Since $A_r f$ is continuous in r, this uncountable supremum can be replaced by a countable supremum from which it follows that Hf is measurable as the pointwise supremum of measurable functions.

{Thm4.10}

Theorem 4.11 (Hardy-Littlewood Maximal Inequality)**.** *There exists a constant C such that for any $f \in L^+ \cap L^1_{loc}$ and for any $\epsilon > 0$,*

$$m\Big(\{x | Hf(x) > \epsilon\}\Big) \leq \frac{C}{\epsilon} ||f||_1.$$

Proof. For brevity fix $A := \{x | Hf(x) > \epsilon\}$ for some $\epsilon > 0$. If $x \in A$ then there exists an $r_x > 0$ such that $A_r f(x) > \epsilon$. We then get a covering

$$A \subseteq \bigcup_{x \in A} B(x, r_x),$$

and by Vitali's Lemma 4.9 there exists disjoint $B_i = B(x_i, r_{x_i})$ such that

$$m\left(\bigsqcup_{i=1}^{k} B_i\right) \geq \frac{1}{C} m(A).$$

Thus,

$$||f||_1 = \int_{\mathbb{R}^n} f dm \geq \int_{\bigsqcup B_i} f dm = \sum_{i=1}^{k} \int_{B_i} f dm \geq \sum_{i=1}^{k} \epsilon m(B_i) \geq \frac{\epsilon}{C} m(A).$$

Hence $m(A) \leq \frac{C}{\epsilon}||f||_1$. □

With the previous results in mind, we are now well-equipped to prove Theorem 4.8.

Proof of Theorem 4.8. It is a simple argument to show that the result holds for continuous functions. Letting $\epsilon > 0$, for a general $f \in L^1_{\text{loc}}$, we may find a continuous function on compact support g such that $||f - g|| < \epsilon$. We set $\varphi := |f - g| \in L^1 \cap L^+$. Let $\alpha > 0$ and consider the set $E_\alpha := \{x \in \mathbb{R}^n \,|H\varphi(x) > \alpha\}$. The Hardy-Littlewood maximal inequality gives $m(E_\alpha) < \frac{C}{\alpha}||\varphi||_1 \leq \frac{C\epsilon}{\alpha}$. We then observe,

$$\begin{aligned} |A_r f(x) - f(x)| &= |A_r(f - g) + A_r g - g + (g - f)|, \\ &\leq A_r\varphi(x) + |A_r g - g|(x) + \varphi(x), \\ &\leq H\varphi(x) + |A_r g - g|(x) + \varphi(x). \end{aligned}$$

By uniform continuity of g, there exists a $\delta > 0$ such that for $r < \delta$ $|A_r g - g|(x) < \alpha$. Set $F_\alpha = \{x \in \mathbb{R}^n \,|\varphi(x) > \alpha\}$ and note by the usual argument $m(F_\alpha) \leq \frac{1}{\alpha}||\varphi||_1 < \frac{\epsilon}{\alpha}$. For $r < \delta$ we note the set,

$$B_{3\alpha,r} := \{x \in \mathbb{R}^n \,||A_r f - f| > 3\alpha\} \subseteq E_\alpha \cup F_\alpha,$$

by the above chain of inequalities. This implies for $x \notin E_\alpha \cup F_\alpha$,

$$\limsup_{r\to 0} |A_r f(x) - f|(x) \leq 3\alpha.$$

In particular for

$$B_{3\alpha} := \{x | \limsup_{r\to 0} |A_r f(x) - f|(x) > 3\alpha\},$$

one has

$$m(B_{3\alpha}) \leq m(E_\alpha \cup F_\alpha) \leq \frac{C+1}{\alpha}\epsilon,$$

for every $\epsilon > 0$. Hence $m(B_{3\alpha}) = 0$ from which we conclude after taking $\alpha \to 0$, the set

$$B := \{x | \limsup_{r\to 0} |A_r f(x) - f|(x) > 0\},$$

has measure 0, completing the proof. □

It turns out this theorem can be strengthened to the following.

{Thm3.11} **Theorem 4.12** (Lebesgue Differentiation Theorem 2)**.** *If $f \in L^1_{loc}$, then for m almost every $x \in \mathbb{R}^n$;*

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm(y) = 0.$$

Proof. Let $c = \{c_i | i \in \mathbb{N}\}$ be any countable dense subset of $\mathbb{R}$. Define $g_i(x) = |f(x) - c_i|$. Applying Theorem 4.8 to each g_i we obtain for $B_i := \{x \in \mathbb{R}^n \, ||A_r g_i - g_i|(x) > 0\}$, $m(B_i) = 0$. Set $G := \mathbb{R}^n \setminus \left(\bigcup_1^\infty B_i \right)$. We note that G is a full measure subset of $\mathbb{R}^n$. We aim to show that the equality holds on all of G. To this end, let $x \in G$, $\epsilon > 0$. By denisty there exists some c_i such that $|f(x) - c_i| < \epsilon$ and since $x \notin B_i$ there exists some r_0 such that for all $r < r_0$, $|A_r g_i(x) - g_i(x)| < \epsilon$. Thus for all $r < r_0$,

$$\begin{aligned} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm(y) &\leq \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - c_i| dm(y) + |f(x) - c_i|, \\ &\leq 2|f(x) c_i| + \epsilon, \\ &\leq 3\epsilon. \end{aligned}$$

□

The first two versions of the Lebesgue differentiation theorem have assumed we were averaging values of smaller and smaller closed balls which shrink towards x. We now show that the sets can be much more general and the result will still hold. For $x \in \mathbb{R}^n$ we say the collection of sets $A_r \subseteq B(x,r)$ *shrink nicely* to x as $r \to 0$ if each $A_r \in \mathcal{L}$ and $m(A_r) > \alpha m(B(x,r)) \geq \alpha' r^n$, for some α' independent r. Interestingly enough we remark that these sets need not contain x. The final version of the Lebesgue differentiation theorem states that this notion of shrinking nicely is precisely the notion needed to guarantee the desired conclusion.

{Thm3.12}

Theorem 4.13 (Lebesgue Differentiaion Theorem)**.** *Given $f \in L^1_{loc}$, for almost every $x \in \mathbb{R}^n$,*

$$\lim_{r \to 0} \frac{1}{m(A_{x,r})} \int_{A_{x,r}} f dm = f(x),$$

as long as $A_{x,r}$ shrink nicely to x.

Proof. The second version of the Lebesgue differentiation theorem 4.12 gives the existence of $L_f \in \mathcal{L}$ with $m(L_f^c) = 0$ and for $x \in L_f$, one has

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm(y) = 0.$$

Hence for $x \in L_f$,

$$\begin{aligned} \frac{1}{m(A_{x,r})} \int_{A_{x,r}} |f(y) - f(x)| dm(y) &\leq \frac{1}{m(A_{x,r})} \int_{B_{x,r}} |f(y) - f(x)| dm(y), \\ &\leq \frac{1}{\alpha' m(B(x,r))} \int_{B_{x,r}} |f(y) - f(x)| dm(y). \end{aligned}$$

With the last term above tending towards 0 as $r \to 0$ we conclude the result. □

5. L^p SPACES

5.1. **First Definitions and Results.** Given $1 < p < \infty$ $(X, \mathcal{M}, \mu)$ a measure space, we define

$$\mathcal{L}^p := \{f : X \to \mathbb{R} \,|\, \int_X |f(x)|^p d\mu < \infty\},$$

where implicitly we are assuming f is measurable. $L^p(X, \mathcal{M}, \mu)$ is then equivalence classes of $\mathcal{L}^p$ under the relation that $f \sim g$ is $f = g$ μ almost everywhere. We define

$$||f||_p = \left(\int_X |f(x)|^p d\mu \right)^{1/p}.$$

We now work towards proving that $|| \cdot ||_p$ defines a norm on L^p and that this norm makes L^p into a Banach space.

{Thm4.1}

Theorem 5.1 (Hölder Inequality). *Let* $1 < p, q < \infty$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then for* $f \in L^p(X, \mathcal{M}, \mu), g \in L^q(X, \mathcal{M}, \mu)$, $fg \in L^1$ *and*

$$||fg||_1 \leq ||f||_p ||g||_q.$$

Before proving this we simply remark that for any $0 \leq \lambda \leq 1$, $a, b > 0$ one has $a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b$ with equality if and only if $a = b$.

Proof. If $||f||_p, ||g||_q$ are either 0 or ∞ the result is trivial and hence we assume both are finite and non-zero. Since $||cf||_p = |c| ||f||_p$ for any $c \in \mathbb{R}$ and likewise for g, it sufficies to prove the result for the case where $||f||_p = ||g||_q = 1$. Using the remark made before the proof, we note for $\lambda = \frac{1}{p}, 1 - \lambda = \frac{1}{q}$,

$$|f(x)g(x)| = \Big(|f(x)|^p\Big)^\lambda \Big(|g(x)|^q\Big)^{1-\lambda} \leq \lambda |f(x)|^p + (1-\lambda)|g(x)|^q.$$

Thus,

$$||fg||_1 \leq \lambda \int |f|^p + (1-\lambda) \int |g|^q = \lambda ||f||_p^p + (1-\lambda)||g||_q^q = 1$$

□

{Thm4.2}

Theorem 5.2 (Minkowski's Inequality). *For* f, g *measurable* $1 < p < \infty$, *one has*

$$||f + g||_p \leq ||f||_p + ||g||_p.$$

Moreover equality holds if and only if $f = 0$ *or* $g = 0$ *almost everywhere or there exists a constant* c *such that* $f = cg$ *almost everywhere.*

Proof. We note that

$$\begin{aligned} ||f+g||_p^p &= \int_X |f+g|^p d\mu, \\ &\leq \int_X |f+g||f+g|^{p-1} d\mu, \\ &\leq |f||f+g|^{p-1} + |g||f+g|^{p-1} d\mu. \end{aligned}$$

Setting $|f + g|^{p-1} = h$ the Hölder Inequality gives,

$$\begin{aligned} ||f+g||_p^p &\leq |f||f+g|^{p-1} + |g||f+g|^{p-1} d\mu, \\ &= ||fh||_1 + ||gh||_1 \leq ||f||_p ||h||_q + ||g||_p ||h||_q, \end{aligned}$$

where $q = 1 - \frac{1}{p}$. We then see that

$$||h||_q = \left(\int \Big(|f+g|^{p-1}\Big)^{p/(p-1)} \right)^{1-1/p} = \left(\int \Big(|f+g|^p\Big) \right)^{1-1/p}.$$

Hence

$$||f+g||_p^p \leq (||f||_p + ||g||_p) \left(\int \Big(|f+g|^p\Big) \right)^{1-1/p} = (||f||_p + ||g||_p) ||f+g||_p^{p(1-1/p)}.$$

□

This inequality implies that L^p is a normed vector space with resepct to $||\cdot||_p$. To conclude this section we show that L^p is complete with respect to this norm.

{Thm4.3}

Theorem 5.3. $(L^p, ||\cdot||_p)$ *is a Banach space.*

Before proving this we prove the following proposition.

{Prop4.4}

Proposition 5.4. *If* $g_n \in L^p(X, \mathcal{M}, \mu)$ *with* $\sum_{n=1}^{\infty} ||g||_p < \infty$. *Then there exists* $G \in L^p$ *such that* $\sum_{n=1}^{m} g_n \to G$ *pointwise almost everywhere and in* L^p.

Proof. Let $F_n = |g_1| + \cdots + |g_n|, G_n = g_1 + \cdots + g_n$. By Minkowski's Inequality,

$$||F_n||_p \leq \sum_{i=1}^{n} ||g_i||_p \leq C,$$

where $C = \sum_{n=1}^{\infty} ||g||_p < \infty$. Thus $\int |F_n|^p = ||F_n||_p^p \leq C^p$. Letting $F = \sup F_n$, and noting $F_1 \leq F_2 \leq \dots$, the monotone convergence theorem gives,

$$\int |F|^p = \lim \int |F_n|^p.$$

Since each $\int |F_n|^p \leq C^p$ we conclude that $F \in L^p$. As $F(x) = \sum_{n=1}^{\infty} |g_n(x)|$ we have that this series is finite for almost every $x \in X$, and thus $\sum_{i=1}^{\infty} g_n(x)$ converges for almost every $x \in X$. Hence $G(x) := \sum_{n=1}^{\infty} g_n(x)$ is well-defined almost every and is the limit of the G_n. We note that $G \in L^p$ since $|G_n(x)|^p \to |G(x)|^p$ almost everywhere and for each n $|G_n(x)|^p \leq F(x)^p$. As $|G|^p \leq F^p \in L^1$, by the dominated convergence theorem we have $\int |G_n(x)|^p \to \int |G(x)|^p$ and hence $\int |G|^p < C^p$ giving that indeed $G \in L^p$. Finally, since $G_n 0 G \to 0$ almost everywhere and $|G_n - G|^p \leq |G_n|^p + |G|^p \leq 2F^p \in L^1$ another application of the dominated convergence theorem gives,

$$\int |G_n - G|^p d\mu \to 0.$$

□

Proof of Theorem 5.3. Let $\{f_n\} \subseteq L^p$ be such that for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n, m > N$ one has $||f_n - f_m||_p < \epsilon$. Then we may find an increasing sequence $n_1 < n_2 < \dots$ such that for every $l > m > n_k$ one has $||f_l - f_m||_p < 2^{-k}$. Define $g_k = f_{n_{k+1}} - f_{n_k}$, $g_0 = f_{n_1}$. Then $\sum ||g_k||_p < \sum 2^{-k} < \infty$ and hence by the previous proposition we have $f = \sum_{k=1}^{\infty} g_k$ is well defined almost everywhere and this sum also converges to f in L^p. Thus $f_{n_k} \to f$ almost everywhere and in L^p. We claim then that $f_n \to f$ in L^p. To this end let $\epsilon > 0$ and choose k large enough so that $||f_{n_k} - f||_p < \epsilon/2$ and for any $n > n_k$ $||f_n - f_{n_k}||_p < \epsilon/2$. Then for $n > n_k$,

$$||f_n - f||_p = ||f_n - f_{n_k} + f_{n_k} - f||_p \leq ||f_n - f_{n_k}||_p + ||f_{n_k} - f||_p \leq \epsilon.$$

□