NOTES ON COMPLEX ANALYSIS

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Contents

These notes originated as a brief overview of undergraduate Complex Analysis but are currently in the process of being expanded into notes following a graduate course in Complex Analysis.

1. Analytic Functions

1.1. **Limits of Complex Valued Functions.** We discuss first the notion of the limit for maps $f : \mathbb{C} \to \mathbb{C}$. Denoting $|z| = z\overline{z}$, we obtain a norm on $\mathbb{C}$ and thus a metric space, and hence inherit the notion of the limit from arbitrary metric spaces.

In particular, given a map $f : \mathbb{C} \to \mathbb{C}$ we say

$$\lim_{z \to z_0} f(z) = w,$$

if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|z - z_0| < \delta$ implies

$$|f(z) - w| < \epsilon .$$

Writing $z \in \mathbb{C}$ as $z = x + iy$, we can write f as

$$f(z) = u(x, y) + iv(x, y),$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$.

Theorem 1.1. *Suppose that* $f(z) = u(x, y) + iv(x, y)$, *and* $z_0 = x_0 + iy_0$. *If*

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0, \qquad \lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0,$$

then

$$\lim_{z \to z_0} f(z) = u_0 + iv_0.$$

Moreover the converse is also true.

The proof of this theorem is a basic result of limits, and as such we omit it. Nevertheless this and the following theorem will be crucial in justifying further results.

Theorem 1.2. *Suppose that* $f, F : \mathbb{C} \to \mathbb{C}$ *satisfy*

$$\lim_{z \to z_0} f(z) = w_0, \qquad \lim_{z \to z_0} F(z) = W_0.$$

Then each of the following hold

- $\lim_{z \to z_0} f(z) + F(z) = w_0 + W_0$,
- $\lim_{z \to z_0} f(z)F(z) = w_0 W_0$,
- *if* $W_0 \neq 0$, *then* $\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}$.

Proof. While we will not provide justification for each of the above, we remark here that Theorem 1.1 can be used to quickly justify each claim. We demonstrate on the first item above. By Theorem 1.1 we have

$$\lim_{z \to z_0} f(z)+F(z) = \lim_{(x,y) \to (x_0,y_0)} (u_f(x, y)+u_F(x, y))+i \lim_{(x,y) \to (x_0,y_0)} (v_f(x, y)+v_F(x, y)).$$

By properties of real-valued limits we then have the above is precisely

$$\lim_{(x,y) \to (x_0,y_0)} u_f(x, y)+i \lim_{(x,y) \to (x_0,y_0)} v_f(x, y)+ \lim_{(x,y) \to (x_0,y_0)} u_F(x, y)+i \lim_{(x,y) \to (x_0,y_0)} v_F(x, y),$$

which by assumption is $w_0 + W_0$, again by Theorem 1.1 □

1.2. **Continuity.** With the notion of the limit defined, we say a function $f : \mathbb{C} \to \mathbb{C}$ is continuous at the point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0),$$

exactly as defined in Real Analysis. The general theory of maps between metric spaces gives the composition of any two continuous functions is also continuous. Moreover an immediate consequence of Theorem 1.1 is the following.

Theorem 1.3. *Let* $f(z) = u(x, y)+iv(x, y)$ *where* $z = x+iy$. *Then* f *is continuous if and only if* u, v *are continuous maps* $\mathbb{R}^2 \to \mathbb{R}$.

Note that the topology on $\mathbb{C}$ is identical to the topology on $\mathbb{R}^2$. Hence compact sets in $\mathbb{C}$ are precisely those sets which are closed and bounded. Since images of compact sets under continuous mappings are compact, we see that the image of any compact set under some continuous map $f : \mathbb{C} \to \mathbb{C}$ is bounded in modulus.

1.3. **Complex Differentiation.** We say a function $f : \mathbb{C} \to \mathbb{C}$ is differentiable at the point z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

exists. In that case, we denote this limit by $f'(z_0)$ and refer to it as the derivative of f at z_0. We now present a necessary, although not sufficient condition, for a complex valued function to be differentiable, known as the Cauchy-Riemann Equations.

Theorem 1.4. *Let $f(z) = u(x, y) + iv(x, y)$ and suppose f is differentiable at $z_0 = x_0 + iy_0$. Then the first-order partial deriviatives, u_x, u_y, v_x, v_y all exist and satisfy*

$$u_x = v_y, \quad u_y = -v_x.$$

Moreover,

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

Proof. For brevity, write $\Delta z = \Delta x + i\Delta y, \Delta w = f(z_0 + \Delta z) - f(z_0)$, so that the limit in question becomes

$$\lim_{\Delta x, \Delta y \to 0} \frac{\Delta w}{\Delta x + i\Delta y}.$$

Then we see

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} + i\frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y}.$$

This limit exists and is thus must be the same independent of the manner in which $\Delta x, \Delta y$ approach 0. If we allow $z \to z_0$ along the horizontal line $(x_0 + \Delta x, y_0)$, the above evaluates to

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x} &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}, \\ &= u_x + iv_x. \end{aligned}$$

Likewise, if we allow $z \to z_0$ along the horizontal line $(x_0 + \Delta x, y_0)$ we achieve,

$$\begin{aligned} \lim_{\Delta y \to 0} \frac{\Delta w}{i\Delta y} &= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y}, \\ &= v_y - iu_y. \end{aligned}$$

Therefore $u_x = v_y, u_y = -v_x$, and the first calculation shows

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

□

While the Cauchy-Riemann equations are necessary conditions for a function to complex differentiable, they are not sufficient. In order to conclude differentiabiliity, we must require more.

Theorem 1.5. *Let $f(z) = u(x, y) + iv(x, y)$ be defined throughout some neighborhood of z_0. Further assume that all first order partial derivatives u_x, u_y, v_x, v_y exist and are continuous at (x_0, y_0), and satisfy the Cauchy Riemann equations at (x_0, y_0). Then $f'(z_0)$ exists.*

Proof. Again, for brevity we will write $\Delta z = \Delta x + i\Delta y, \Delta w = f(z_0 + \Delta z) - f(z_0)$, in an effort to evaluate

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}.$$

Since all first order partial deriviatves exist and are continuous at (x_0, y_0), u, v are differentiable with their derivatives being their respective Jacobian matrices. This allows us to write

$$\begin{aligned} u(x,y) &= u(x_0,y_0) + u_x(x_0,y_0)\Delta x + u_y(x_0,y_0)\Delta y + \epsilon_1\, \Delta x + \epsilon_2\, \Delta y, \\ v(x,y) &= v(x_0,y_0) + v_x(x_0,y_0)\Delta x + v_y(x_0,y_0)\Delta y + \epsilon_3\, \Delta x + \epsilon_4\, \Delta y \end{aligned}$$

where $\epsilon_1, \ldots, \epsilon_4$ tend to 0 as $\Delta x, \Delta y \to 0$. The above, along with the Cauchy-Rieman equations gives

$$\frac{\Delta w}{\Delta z} = u_x(x_0,y_0) + iv_x(x_0,y_0) + (\epsilon_1 + i\,\epsilon_3)\frac{\Delta x}{\Delta z} + (\epsilon_2 + i\,\epsilon_4)\frac{\Delta y}{\Delta z}.$$

It is easy to see the error terms tend to 0 as Δz approaches 0, giving the desired result and the resulting equality. □

1.4. **Polar Coordinates.** It is sometimes more beneficial to consider a function $f : \mathbb{C} \to \mathbb{C}$ in polar coordinates. For the time being, we take for granted Euler's identity,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Then given a point $z \in \mathbb{C}$ with magnitude r angle above the horizontal θ, we may write

$$z = re^{i\theta}.$$

Given a function $f : \mathbb{C} \to \mathbb{C}$, written as $f(x,y) = u(x,y) + iv(x,y)$ together with the relationship for any point in $\mathbb{C}$ that

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we can then compute the partial derivatives of u, v with respect to r, θ. Namely, using the chain rule we see

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r},$$

and similarly for the other partial derivatives. Upon carrying out these calculations we arrive at

$$\begin{aligned} u_r &= u_x \cos\theta + u_y \sin\theta, \quad u_\theta = -u_x r\sin\theta + u_y r\cos\theta, \\ v_r &= v_x \cos\theta + v_y \sin\theta, \quad v_\theta = -v_x r\sin\theta + v_y r\cos\theta. \end{aligned}$$

Then, if the Cauchy-Riemann equations are stasfied, their polar analogue simplifies to

$$ru_r = v_\theta, \quad u_\theta = -rv_r.$$

Likewise if the above equations hold, so too do the Cauchy-Riemann equations and thus the above is simply another equivalent condition of the Cauchy-Riemann equations.

1.5. **Analytic Functions.** We use this final section to introduce further terminology for differentiable functions of a complex variable, as well as prove the analogues of some theorems from basic calculus. Given an open set $A \subseteq \mathbb{C}$, we say the function $f : A \to \mathbb{C}$ is analytic on A if it is differentiable at every point in A. We say f is analytic at the point z_0 if there exists some open neighborhood of z_0 on which f is analytic. A function that is analytic on all of $\mathbb{C}$ is said to be entire. For the exact same reasons as in basic calculus, differentiability on some open set A implies continuity on that set. We also suggest the reader verify that sum, differences, and products of analytic functions are also analytic, with their derivatives following the same formulas from functions of a real variable. The same can be said for the composition of two analytic functions.

Theorem 1.6. *Suppose f is analytic on some open connected domain D with $f'(z) = 0$ at all $z \in D$. Then f is constant on D.*

Proof. Given a point $z \in D$, we know

$$f'(z) = u_x + iv_x.$$

and hence $u_x, v_x = 0$ at x, y where $z = x + iy$. Moreover the Cauchy-Riemann equations give that indeed all first order partials evaluate to 0 at x, y. To conclude, we show that f is constant along any line segment starting at z. (This suffices to conclude since D is connected). To that end, we know from basic calculus that the directional derivative along some line segment L is given by

$$\nabla u \cdot \vec{u},$$

where $\vec{u}$ is the unit vector in the direction of L. Evidently $\nabla u = \vec{0}$ at every point in D and hence the directional derivativen of both u and v along any line-segment is in fact 0. From this we conclude u, v are constant on any line segment, from the multivariable mean-value theorem. □

Proposition 1.7. *Suppose that f and its conjugate $\overline{f}$ are analytic throughout the open connected domain D. Then f is constant throughout D.*

Proof. Write $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$, and note by analyticity the Cauchy-Riemann equations,

$$u_x = v_y, \quad u_y = -v_x,$$

are satisfied throughout D. Moreover since $\overline{f}(z) = u(x, y) - iv(x, y)$, is also analytic, the Cauchy-Riemann equations give

$$u_x = -v_y, \quad u_y = v_x.$$

From this we obtain $u_x, u_y = 0$ throughout D and since

$$f'(z) = u_x + iu_y,$$

we see that $f' \equiv 0$ on D. By Theorem 1.6 the result is therefore obtained. □

Proposition 1.8. *Suppose that f is analytic throughout an open connected domain D with constant modulus $|f(z)| = c$ throughout D. Then f is constant on D.*

Proof. If $|f(z)| = 0$ throughout D then $f(z) = 0$ throughout D. If $|f(z)| > 0$ throughout D then we have

$$f(z)\overline{f}(z) = c > 0,$$

throughout D and thus $f(z)$ is non-zero throughout D. Hence we may write

$$\overline{f}(z) = \frac{c}{f(z)},$$

from which we see that $\overline{f}$ is analytic on D. Hence from Proposition 1.7 f is constant on D. □

2. Integrals

2.1. Maps $\mathbb{R} \to \mathbb{C}$. Let $w : \mathbb{R} \to \mathbb{C}$, which we will sometimes simply write $w(t)$ or expand to

$$w(t) = u(t) + iv(t).$$

Again, as $\mathbb{R}$ and $\mathbb{C}$ are both metric spaces, the ordinary notion of limits of such functions apply, and in particular the notion of continuity of such functions is just as the notion of continuity of maps between abstract metric spaces. For differntiability we say the function w is differntiable at t_0 if the limit

$$\lim_{h\to 0} \frac{w(t_0 + h) - w(t_0)}{h},$$

exists and denote this limit by $w'(t_0)$. Note

$$\lim_{h\to 0} \frac{w(t_0 + h) - w(t_0)}{h} = \lim_{h\to 0} \frac{u(t_0 + h) - u(t_0)}{h} + i \lim_{h\to 0} \frac{v(t_0 + h) - v(t_0)}{h},$$

from which one easily sees that provided both limits on the right exists,

$$w'(t_0) = u'(t_0) + iv'(t_0).$$

Given such a map w, there is also a similarly intuitive defintion of the integral of w. Let $w : [a, b] \to \mathbb{C}$, $w(t) = u(t) + iv(t)$ be given , where u, v are both bounded on $[a, b]$. Then we define the definite integral of w over $[a, b]$ to be given by

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt,$$

provided both integrals on the right exist.

Recall from real analysis that any function which is continuous at all but finitely many points on the interval $[a, b]$ is integrable. Then, clearly from the above definition, the same also applies in this case. For brevity, we will call a function who is continuous at all but finitely many points at some interval piecewise continuous on that interval. We now examine how the fundamental theorem of calculus applies to integrals of the above type. Suppose the functions

$$w(t) = u(t) + iv(t), \quad W(t) = U(t) + iV(t),$$

are continuous and satisfy $W'(t) = w(t)$. In particular this gives continuity of u, v, U, V and $U'(t) = u(t), V'(t) = v(t)$. Hence two applications of the fundamental theorem of calculus gives

$$\int_a^b w(t)dt = W(b) - W(a).$$

2.2. **Path Integrals.** In order to define integrals of functions $f : \mathbb{C} \to \mathbb{C}$, we must recall the notion of continuous paths. We define a continuous path in $\mathbb{C}$ to be a continuous map $\alpha : [a, b] \to \mathbb{C}$ where $[a, b]$ is some compact interval in $\mathbb{R}$. Such a path is called simple if $\alpha(t_1) \neq \alpha(t_2)$ for any $t_1 \neq t_2$. Moreover if $\alpha(a) = \alpha(b)$ we say α is closed.

We will often write these paths as $z(t) = x(t) + iy(t)$. Supposing both x, y are differentiable functions of t with continuous derivatives, the real-valued function

$$|z'(t)| = \sqrt{x'(t)^2 + y'(t)^2},$$

is continuous and therefore integrable. Moreover, from basic calculus, the arc length of the path is precisely

$$L = \int_a^b |z'(t)| dt.$$

With this in mind we can now define the integral of a map $f : \mathbb{C} \to \mathbb{C}$ along some path C. Let the path C be parametrized by $z(t)$, and suppose that f is piecewise continuous along $z(t)$. The path integral is then defined as

$$\int_C f dz = \int_a^b f(z(t)) z'(t) dt.$$

For example, we evaluate the contour integral

$$\int_C \frac{dz}{z},$$

where C denotes the top half of unit circle in C. C can be parametrized by

$$z(\theta) = e^{i\theta},$$

where $0 \leq \theta \leq \pi$, which we see

$$\int_C \frac{dz}{z} = \int_0^\pi \frac{ie^{i\theta}}{e^{i\theta}} = \pi i.$$

We conclude this section with a theorem for an upper bound of the modulus of a given path integral. First, we need the following lemma

Lemma 2.1. *If $w : [a, b] \to \mathbb{C}$ is a piecewise continuous function on $[a, b]$, then*

$$\left| \int_a^b w(t) dt \right| \leq \int_a^b |w(t)| dt.$$

Proof. Piecewise continuity ensures existence of the integral in question and hence we may write

$$\int_a^b w(t) dt = r_0 e^{i\theta_0}.$$

Rewriting, using the fact that $1/e^{i\theta_0} = e^{-i\theta_0}$, we obtain

$$r_0 = \int_a^b e^{-i\theta_0} w(t) dt.$$

The left hand side is real and thus so is the integral on the right. Hence

$$r_0 = \Re \int_a^b e^{-i\theta_0} w(t) dt = \int_a^b \Re[e^{-i\theta_0} w(t)] dt.$$

To conclude, simply note

$$\Re[e^{-i\theta_0}w(t)] \leq |e^{-i\theta_0}w(t)| \leq |w(t)|,$$

and hence

$$r_0 \leq \int_a^b |w(t)|dt,$$

as claimed. □

Theorem 2.2. *Let C denote a path iof length L in $\mathbb{C}$. Suppose that $f(z)$ is piecewise continuous on C. If $M > 0$ is such that*

$$|f(z)| \leq M,$$

on all of C, then

$$\left|\int_C f(z)dz\right| \leq ML.$$

Proof. Let $z(t)$ parametrize C over the interval $[a, b]$. Then using Lemma 2.1, we see

$$\left|\int_C f(z)dz\right| = \left|\int_a^b f(z(t))z'(t)dt\right| \leq \int_a^b |f(z(t))z'(t)|.$$

Since the integrand on the right is bounded by $M|z'(t)|$, and is a simply an integral of a real valued function, we obtain

$$\left|\int_C f(z)dz\right| \leq M \int_a^b |z'(t)|dt,$$

whence the result follows upon noting that the integral on the right is precisely L. □

It is worth noting that a bound M as stated in the theorem will always exist. This is because $[a, b]$ is a compact set and thus $f(z(t))$ for all $t \in [a, b]$ is also compact.

2.3. **The Closed Curve Theorem.** In the previous section, we defined integrals along a path in $\mathbb{C}$. This definition, although independent of the parametrization of the path, was wholly dependent on the choice of path. In some cases, however, path integrals may be completely independent of path, and dependent only on the endpoints of the path.

Theorem 2.3. *Suppose that $f(z)$ is continuous on an open, connected domain D. Then the following are equivalent.*

a) There exists a function $F : D \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in D$.
b) Given two points $z_1, z_2 \in D$ and any path C connecting z_1 to z_2, one has

$$\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1).$$

c) The integral of f along any closed path in D evaluates to 0.

Proof. We start with *a)* implies *b)*. Suppose C is a smooth arc with parametrization $z(t)$ over $[a, b]$ such that $z(a) = z_1, z(b) = z_2$. Then via the chain rule

$$\frac{d}{dt}[F(z(t))] = F'(z(t))z'(t) = f(z(t))z'(t).$$

In section 2.1 we showed the fundamental theorem of calculus applies to integrals of functions $f : \mathbb{R} \to \mathbb{C}$, and hence

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t) = F(z_2) - F(z_1),$$

as desired. Moreover, if C consists of a finte number of smooth arcs, one easily sees that from the above, the result follows.

For *b)* implies *c)*, let C be a closed path and let $z_1, z_2 \in C$ be any two arbitary points. Let C_1, C_2 be two paths which both start at z_1 and end at z_2 so that $C = C_1 - C_2$. By independence of path,

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz,$$

from which it follows that

$$\int_C f(z)dz = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0,$$

as desired.

Finally we show *c)* implies *a)*. We first show integration is independent of path in D assuming integrals along closed contours are 0. To that end, let C_1, C_2 denote any two paths in D connecting z_1 to z_2. Then evidently $C = C_1 - C_2$ is a closed path giving

$$\int_C f(z)dz = 0,$$

and therefore

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Hence we may define the function

$$F(z) = \int_{z_0}^{z} f(z)dz,$$

unambiguously, for some fixed base point $z_0 \in D$. Observe then that for any $z \in D$,

$$F(z + \Delta z) - F(z) = \int_z^{z+\Delta z} f(s)ds.$$

We may write

$$f(z) = \frac{1}{\Delta z} \int_z^{z+\Delta z} f(z)ds,$$

and thus it follows that

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_z^{z+\Delta z} [f(s) - f(z)]ds.$$

By continuity of f, for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|s - z| < \delta$ gives $|f(s) - f(z)| \leq \epsilon$. Hence for s satisfying

$$|s - z| < \delta,$$

we see

$$\frac{1}{\Delta z} \int_z^{z+\Delta z} [f(s) - f(z)]ds \leq \epsilon,$$

by Theorem 2.2. Thus

$$F'(z) = f(z).$$

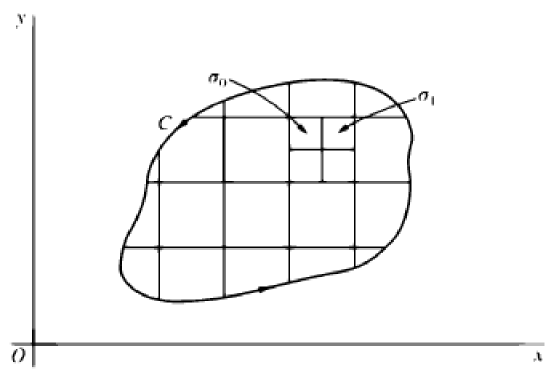
□

To conclude this section, we present what is known as the *Cauchy-Goursat Theorem* which shows that analyticity on a domain D is enough to reach the conclusions of the closed curve theorem.

Theorem 2.4. *If f is analytic on all points interior to and on a simple closed path C then*

$$\int_C f dz = 0.$$

Proof. We begin by partitioning the interior of the curve C into squares by drawing equally spaced horizontal and vertical lines. By doing so, we partition the interior into finitely many squares and finitely many partial squares whose boundary partially consists of the curve C. This is demonstrated in the figure below. With this



partition in mind we have the following lemma.

Lemma 2.5. *Let R denote the region interior to the simple closed curve C together with the points belonging to C itself. For any $\epsilon > 0$, the region can be covered with a finite number of squares and partial sqaures, as described above, such that in each square there exists a fixed point z_j, for which the inequality*

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \epsilon,$$

holds for all z in that square.

As illustrated in Figure 2.4 given a covering which contains a square such that no fixed point exists satisfying the inequality of the lemma, we may divide such a square into four smaller squares by drawing lines connecting the midpoints of its opposite sides. Reaching for a contradiction, we therefore assume given a covering of R, there is a square such that the needed fixed points are not obtained after any finite number of subdivisions of the original square. Let σ_0 denote the original square. By assumption after subdividing σ_0 into four smaller squares, one of the smaller squares must still lack the desired fixed point, which we denote by σ_1. Continuing iteratively, we obtain a sequence of nested sets

$$\sigma_0, \sigma_1, \sigma_2, \ldots,$$

Via the nested set property, each of these sets contain a common point which we denote by z_0, and each individual square σ_k contains points of R different from z_0. Moreover the size of these squares are decreasing and hence any δ-neighborhood of z_0 will contain squares of small enough area. Hence every δ-neighborhood of z_0 contains points of R different from z_0 and thus z_0 is an accumulation point of R. Since R is closed, we have that $z_0 \in R$.

Since f is analytic throughout R, it is in particular analytic at z_0, from which we obtain for $\epsilon > 0$ there is a $\delta > 0$ such that $|z - z_0| < \delta$ implies

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \epsilon\,.$$

But since there is a K large enough so that σ_K lies entirely in the δ-neighborhood of z_0, we have that z_0 is a fixed point satisfying the desired inequality of the lemma. This contradicts the manner in which the σ_j were chosen, completing the proof of the lemma.

Now, let $\epsilon > 0$ and cover R by squares so that the inequality in Lemma **??** is satisfied in each square. On the jth square define the function $\delta_j(z)$ by $\delta_j(z_j) = 0$, and

$$\delta_j(z) = \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j).$$

By the Lemma, $|\delta_j(z)| < \epsilon$ for all z in the jth square and by analyticity this function is continuous.

Let C_j denote the positiviely oriented boundaries of the squares and partial squares covering R. Using $\delta_j(z)$ we see that the value of f at a point z on any C_j can be written

$$f(z) = f(z_j) - z_j f'(z_j) + f'(z_j)z + (z - z_j)\delta_j(z).$$

Hence

$$\int_{C_j} f(z)dz = [f(z_j) - z_j f'(z_j)] \int_{C_j} dz + f'(z_j) \int_{C_j} zdz + \int_{C_j} (z - z_j)\delta_j(z)dz.$$

Theorem 2.3 gives that $\int_{C_j} dz, \int_{C_j} zdz = 0$, both 1 and z have antiderivatives on all of $\mathbb{C}$. Thus the above integral simplifies to

$$\int_{C_j} f(z)dz = \int_{C_j} (z - z_j)\delta_j(z)dz.$$

Now, summing all integrals over the C_j, we note that

$$\sum_{j=1}^{n} \int_{C_j} f(z)dz = \int_C f(z)dz,$$

since each of side of a square will appear twice in the left hand sum, and in opposite directions. Hence all integrals over sides of squares will cancel to 0, leaving only the integrals along the closed path C. Thus

$$\int_C f(z)dz = \sum_{j=1}^{n} \int_{C_j} (z - z_j)\delta_j(z)dz.$$

The triangle ineqaulity, together with Theorem 2.2 then give

$$\left|\int_C f(z)dz\right| \le \sum_{j=1}^{n}\int_{C_j} |(z-z_j)\delta_j(z)|\, dz.$$

Let s_j denote the side length of the jth square. Then we have

$$|z-z_j| \le \sqrt{2}s_j,$$

since both z, z_j lie in the jth square. Moreover, by the lemma $|\delta_j(z)| \le \epsilon$ for all z in the jth square. Hence

$$|(z-z_j)\delta_j(z)| \le \sqrt{2}s_j\,\epsilon\,.$$

As for the length of C_j, if C_j is a complete square it is $4s_j$ and if C_j is a partial square, its length does note exceed $4s_j + L_j$ where L_j denotes the length of the segment of C which makes up the boundary of the partial square. If we let A_j denote the area of the jth full square, we otbain in either case that

$$\left|\int_{C_j}(z-z_j)\delta(z)dz\right| < \sqrt{2}s_j\,\epsilon(4s_j + L_j) < 4\sqrt{2}A_j\,\epsilon + \sqrt{2}SL_j\,\epsilon,$$

where S denote the side length of some square entirely enclosing the contour C and the squares of the intial covering. Since the sum of all A_j cannot exceed S^2, we then see that altogether

$$\left|\int_C f(z)dz\right| \le (4\sqrt{2}S^2 + \sqrt{2}SL)\,\epsilon,$$

and letting $\epsilon \to 0$ we see that the integral in question indeed evaluates to 0, completing the proof. □

2.4. **Deformation of Contours.** As a consequence of the previous two sections, we now present a theorem which will allow us to deform one contour into another, and still maintain the same value of the integral.

Theorem 2.6. *Suppose that C is a simple closed path oriented in the counterclockwise direction and $C_k, k = 1, \ldots, n$ are simple closed paths interior to C, all oriented clockwise, that are disjoint and whose interiors have empty intersection. Then if f is analytic on all of these paths and throughout the multply connected domains formed by the points interior to C and exterior to each C_k, then*

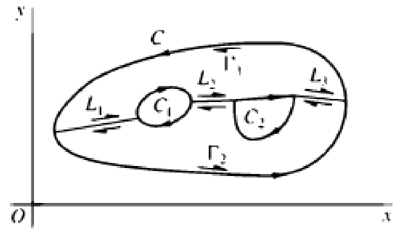
$$\int_C f(z)dz + \sum_{k=1}^{n}\int_{C_k} f(z)dz = 0.$$

Proof. For the proof, we assume there exist only two interior paths C_1, C_2 as a simple induction proves the entire theorem. In this case we form lines L_1, L_2, L_3, connecting C to C_1, C_1 to C_2 and C_2 to C respectively as drawn in Figure 2. From this we obtain two simple closed contours Γ_1, Γ_2 and via the Cauchy-Gorsat theorem we see

$$\int_{\Gamma_1} f dz + \int_{\Gamma_2} f dz = 0.‘$$

After the cancellation obtained from integrating along L_1, L_2, L_3 in oppsoite directions, only the integrals along C, C_1, C_2 remain, giving the result. □

The following corollary is what is known as the principle of deformation of paths.



Corollary 2.7. *Suppose C_1, C_2 are two positively oriented simple closed paths with C_2 interior to C_1. If f is analytic on both paths and at every point in between the two paths, then*

$$\int_{C_1} f dz = \int_{C_2} f dz.$$

2.5. **The Cauchy Integral Formula.**

Theorem 2.8. *Let f be analytic everywhere inside and on a simple closed path C, oriented clockwise. If z_0 is any point interior to C, then*

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0} dz.$$

Proof. Let C_ρ denote a positively oriented circle of radius ρ centered at z_0 where ρ is small enough so that C_ρ is interior to C. Since f analytic, $f(z)/z - z_0$ is analytic on C, C_ρ and between the two curves, and so via deformation of paths we see

$$\int_C \frac{f(z)}{z - z_0} = \int_{C_\rho} \frac{f(z)}{z - z_0}.$$

Therefore

$$\int_C \frac{f(z)}{z - z_0} - f(z_0)\int_{C_\rho} \frac{dz}{z - z_0} = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz.$$

Since

$$\int_{C_\rho} \frac{dz}{z - z_0} = 2\pi i,$$

we obtain

$$\int_C \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz.$$

The result will follow once we show the integral on the right evaluates to 0. Let $\epsilon > 0$ and by continuity of f at z_0 let $\delta > 0$ be such that $|z - z_0| < \delta$ implies that

$$|f(z) - f(z_0)| < \epsilon\,.$$

Replacing ρ by something smaller than δ if necessary, we then see that

$$\left| \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \frac{\epsilon}{\rho} 2\pi\rho = 2\pi\,\epsilon\,.$$

Taking $\epsilon \to 0$ the result is then obtained. □

In fact, more is true as the following theorem shows.

Theorem 2.9. *Let f be analytic inside and on a simple closed contour C, oriented positively. If z_0 is any point interior to C, then*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(s)ds}{(s-z_0)^{n+1}}.$$

Proof. For this proof we only show the result for the first derivative of f and remark that induction can give the formulation for all other derivatives. The Cauchy Integral Formula as in Theorem 2.8 gives that

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{s-z_0},$$

for any z_0 interior to C. Fixing z in the interior of C, and taking $|\Delta z|$ small enough so that $z+\Delta z$ remains interior to C, we obtain

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right)\frac{f(s)}{\Delta z}ds,$$

and therefore

$$\frac{f(z+\Delta z)-f(z)}{\Delta z}\frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z-\Delta z)(s-z)}.$$

Upon noting,

$$\frac{1}{(s-z-\Delta z)(s-z)} = \frac{1}{(s-z)^2} + \frac{\Delta z}{(s-z-\Delta z)(s-z)^2},$$

we see

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)^2} = \frac{1}{2\pi i}\int_C \frac{\Delta z f(s)ds}{(s-z-\Delta z)(s-z)^2}.$$

If we let d denote the minimum distance between z and a point s on the curve C, and M denote the maximum value of f obtained on C, we observe that

$$|s-z-\Delta z| \geq ||s-z|-|\Delta z|| = d-|\Delta z| > 0.$$

And therefore

$$\left|\int_C \frac{\Delta z f(s)ds}{(s-z-\Delta z)(s-z)^2}\right| \leq \frac{|\Delta z|M}{(d-|\Delta z|)d^2}L,$$

where L denotes the length of C. Letting $\Delta z \to 0$ we see this integral evaluates to 0, proving the result. □

2.6. **Consequences of the Cauchy Integral Formula.** In this final section we detail many consequences of the generalized form of the Cauchy Integral Formula, with the two most notable being Liouville's Theorem and the Maximum Modulus Principle.

Theorem 2.10. *If a function f is analytic at some point, then its derivatvies of all orders are analytic there too.*

Proof. Suppose that f is analytic at z_0 so that there exists an ϵ neighborhood around z_0 on which f is analytic. Let C_0 denote the positively orientied circle of radius $\epsilon/2$ centered at z_0. By Theorem 2.9,

$$f''(z) = \frac{1}{\pi i}\int_{C_0} \frac{f(s)ds}{(s-z)^3},$$

for each z interior to C_0. In particular this implies analyticity of f' at z_0. An identical argument applied to f' will then show f'' is analytic and continuing inductively we obtain the result. □

This theorem is remarkable and stunning contrast to functions of real variables, where existence of any number of derivatives is possible without the existence of a derivative of the next order. The Cauchy Integral Formula also implies the following converse of the Cauchy-Gorsat Theorem.

Theorem 2.11. *Suppose f is continuous throughout the open, connected domain D and for any closed curve in D,*

$$\int_C f dz = 0.$$

Then f is analytic throughout D.

Proof. By Theorem 2.3 our assumptions imply f has an antiderivative F on all of D. F is therefore analytic with derivative f and by Theorem 2.10, f is analytic on D. □

We now present what is often referred to as *Liouville's Theorem*

Theorem 2.12. *Any bounded entire function is constant.*

Proof. Let $z_0 \in \mathbb{C}$ be arbitrary, C_R denote the circle of radius R centered at z_0 and M be such that

$$|f(z)| \leq M,$$

for all $z \in \mathbb{C}$. The Cauchy Integral Formula gives

$$f'(z_0) = \frac{1}{2\pi i} \int_{C_R} \frac{f(s)ds}{(s-z)^2}.$$

Using our standard bounds on path integrals we obtain

$$|f'(z_0)| \leq \frac{M}{R}.$$

Since this holds for any R we may take $R \to \infty$ after which we obtain that

$$|f'(z_0)| = 0.$$

Since f has zero derivative throughout all of $\mathbb{C}$ we conclude it is constant. □

We provide a brief lemma aiming towards a theorem we will prove later, before concluding this section by discussing the Schwarz reflection principle.

Lemma 2.13. *Suppose $|f(z)| \leq |f(z_0)|$ at each point z in some ϵ neighborhood of z_0, on which f is analytic. Then f takes the constant value $f(z_0)$ throughout the entire neighborhood.*

Proof. Let z_1 be an arbitrary point other than z_0 in the ϵ neighborhood. Denote by ρ the distance between z_0 and z_1 and let C_ρ denote the circle of radius ρ centered at z_0. By the Cauchy Integral Formula

$$f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)dz}{z-z_0}.$$

Parametrizing C_ρ by

$$z(\theta) = z_0 + \rho e^{i\theta},$$

we see that
$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta.$$
Before continuing, we note that the above says when a function is analytic on and within a circle, the value at its center is the arithmetic mean of the values along the circle. This is often referred to as *Gauss' mean value theorem*. From this equation we obtain
$$|f(z_0)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0) + \rho e^{i\theta}|d\theta.$$
On the other hand, since f obtains its maximum modulus at z_0, we find that
$$\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|d\theta \leq \int_0^{2\pi} |f(z_0)|d\theta = 2\pi f(z_0).$$
Hence
$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0) + \rho e^{i\theta})|d\theta,$$
which is equivalent to
$$\int_0^{2\pi} [|f(z_0)| - f(z_0 + \rho e^{i\theta})]d\theta = 0.$$
The above however, is the integral of a continuous function whose values are always greater than or equal to 0. Hence $|f(z_0)| = |f(z_0 + \rho e^{i\theta})|$ for all θ. Now since this holds for any $\rho < \epsilon$, we see that $|f(z)|$ is constant on the entire ϵ neighborhood of z_0. Thus by Proposition 1.8 we see that f is constant throughout the ϵ neighborhood. □

Let $\Omega \subseteq \mathbb{C}$ be a set which is symmetric about the $\mathbb{R}$. In particular this means for any $z \in \Omega, \overline{z} \in \Omega$. We will let Ω^+ denote the subset of Ω which lies in the upper half plane, and similarly Ω^- corresponds to the subset belonging to the lowe half plane. Finally let $I = \Omega \cap \mathbb{R}$.

Theorem 2.14. *Let f^+ and f^- be analytic functions on Ω^+ and Ω^- respectively which extend continuously onto I such that*
$$f^+(x) = f^-(x),$$
for each $x \in I$. Then the function $f : \Omega \to \mathbb{C}$ defined via,
$$f(z) = \begin{cases} f^+(z) & \Im z > 0, \\ f^+(z) = f^-(z) & z \in I, \\ f^-(z) & \Im z < 0, \end{cases}$$
is analytic.

Proof. We need only check that f is analytic at each $z \in I$. For $z \in I$ let D be an open connected domain containing z and contained in Ω. Let C be a closed contour in D. Obviously if C does not intersect I then
$$\int_C f dz = 0,$$
as f^+ and f^- are analytic in Ω^+ and Ω^- respectively. Now if C intersects I at a single point, or along a single connected interval of I, upon raising the curve off of I by say $\epsilon > 0$, we see that $\int_{C_\epsilon} f dz = 0$ and letting $\epsilon \to 0$ we attain $\int_C f dz = 0$.

All other cases may be reduced to the previous two cases and so we conclude by Morera's Theorem that f is analytic in D. □

We now generalize this result to what is commonly referred to as the Schwarz Reflection Principle.

Theorem 2.15. *Suppose f is analytic in Ω^+ and f extends continuously to I such that $f(I) \subseteq \mathbb{R}$. Then there exists an extension $F : \Omega \to \mathbb{C}$ which is analytic on Ω with $F|_{\Omega^+} = f$.*

Proof. For $z \in \Omega^-$ defined,

$$F(z) = \overline{f(\overline{z})}.$$

For $z, z_0 \in \Omega^-$, the power series expansion for f near $\overline{z_0}$ is given by,

$$f(\overline{z}) = \sum_{n=0}^{\infty} a_n (\overline{z} - \overline{z_0})^n,$$

and therefore,

$$F(z) = \sum_{n=0}^{\infty} \overline{a_n} (z - z_0)^n.$$

Thus F is analytic in Ω^- and obviously extends continously to I such that F and f agree on I. Invoking Theorem 2.14 for F and f completes the proof. □

Lemma 2.16. *Suppose $U \subseteq \mathbb{C}$ is simply connected, $f : U \to \mathbb{C}$ is analytic and nowhere zero on U. Then there exists an analytic $g : U \to \mathbb{C} \setminus \{0\}$, such that*

$$g(z)^2 = f(z),$$

for all $z \in U$.

Proof. Since f is nowhere vanishing the function f'/f is analytic on U. Since U is simply connected, f'/f has a primitive in U, say h. Upon fixing some $z_0 \in U$ we can add a constant so that

$$e^{h(z_0)} = f(z).$$

Then for arbitrary $z \in U$,

$$\frac{d}{dz}(f(z)e^{-h(z)}) = 0,$$

and so $f(z)e^{-h(z)}$ is constant on U. In fact by construction $f(z)e^{-h(z)} \equiv 1$ on U. Hence

$$f(z) = e^{h(z)},$$

Taking

$$g(z) = e^{h(z)/2},$$

obtains the result. □

3. Taylor and Laurent Series

3.1. **Taylor Series for Analytic Functions.** Thus far we have established the Cauchy Integral Formula and a few of its stunning consequences, one of which being that a function which is analytic in a domain D, has derivatives of all order in D. Naturally, this leads to a discussion of power zeries for analytic functions.

Theorem 3.1. *Suppose f is analytic at some point $a \in \mathbb{C}$, and thus differentiable on some neighborhood $D_r(a)$. Then for $z \in D_r(a)$,*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n.$$

Proof. By the Cauchy Integral Formula, we know that

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)},$$

where C is the boundary of the disk $D_r(a)$. We then write,

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a - (z-a)} = \frac{1}{\zeta - a}\frac{1}{1 - \frac{z-a}{\zeta-a}}.$$

We now that $|\zeta - a| = r$ since ζ lies on $C_r(a)$, and as z in inside $D_r(a)$, we necessarily have that

$$\left|\frac{z-a}{\zeta-a}\right| < 1.$$

Hence we have the geometric series expression,

$$\frac{1}{1-\frac{z-a}{\zeta-a}} = \sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^n.$$

Moreover this convergence is uniform, so we may exchange the sum and integral as follows,

$$\begin{aligned} f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)} &= \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^n, \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-a)^{n+1}}\right)(z-a)^n. \end{aligned}$$

By the Cauchy Integral Formula for derivatives, this is exactly the desired series. □

We now present an easy lemma regarding power series which will allow us to develop the notion of the radius of convergence.

Lemma 3.2. *Suppose $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges for some $z_0 \neq a$. Then the series converges aboslutely for any z such that $|z-a| \leq |z_0 - a|$, and it converges uniformly for all z such that $|z-a| \leq \delta < |z_0 - a|$ for any $\delta > 0$.*

Proof. By assumption, we may find N so that for all $n \geq N$,

$$|a_n(z_0 - a)^n| \leq 1.$$

Thus there is a constatn K such that for all n,

$$|a_n| \leq \frac{K}{|z_0 - a|^n}.$$

Now, if $|z-a| \leq |z_0 - a|$, we obtain

$$\left|\sum_{n=M}^{N} a_n(z-a)^n\right| \leq \sum_{n=N}^{M} |a_n(z-a)^n|,$$
$$\leq \sum_{n=M}^{N} \frac{K}{|z_0-a|^n}|z-a|^n,$$
$$= \sum_{n=M}^{N} K\left(\frac{|z-a|}{|z_0-a|}\right)^n.$$

Since $\frac{|z-a|}{|z_0-a|} < 1$, we have that $\sum_{n=0}^{\infty} K\left(\frac{|z-a|}{|z_0-a|}\right)^n$ converges and is therefore Cauchy. Hence for suitable M, N, $\left|\sum_{n=M}^{N} a_n(z-a)^n\right|$ is small and thus the series is Cauchy, as desired. □

With this in mind, we define the radius of convergence of the series $\sum a_n(z-a)^n$ to be the largest R for which the series converges for all $z \in \mathbb{C}$ satisfying $|z-a| < R$. We adopt the standard convention that if the series only converges at a then $R = 0$ and if the series converges on all of $\mathbb{C}$ then $R = \infty$. Just as in the case of real variables, we can find the radius of convergence using the so-called root test.

Proposition 3.3. *Let $\sum_{n=0}^{\infty}(z-a)^n$ be a power series with radius of convergence R. Then*

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Proof. For brevity set $\mu = \limsup_{n\to\infty} |a_n|^{1/n}$. Consider first the case where $0 < \mu < \infty$ and fix $\epsilon > 0$. Then by definition of the lim sup there exists an N_0 such that for all $n > N_0$ one has

$$|a_n|^{1/n} < \mu + \epsilon,$$

and hence $|a_n| < (\mu+\epsilon)^n$. We may then find a constant K such that for all $n \in \mathbb{N}$ we have

$$|a_n| < K(\mu+\epsilon)^n.$$

With this observation, note

$$\left|\sum_{n=0}^{\infty} a_n(z-a)^n\right| \leq K\sum_{n=0}^{\infty}(\mu+\epsilon)^n|z-a|^n.$$

The right hand side is a geometric series and thus converges when $(\mu+\epsilon)|z-a| < 1$, or equivalently when

$$|z-a| < \frac{1}{\mu+\epsilon}.$$

Thus the series converges for $|z-a| < \frac{1}{\mu}$.

To show the series diverges for $|z-a| > \frac{1}{\mu}$, by the definition of the lim sup there exists a subsequence $|a_{n_k}|^{1/n_k} \to \mu$. Thus there is an N_1 such that for all $n_k > N_1$ we have $|a_{n_k}|^{1/n_k} > \mu - \epsilon$, and thus

$$|a_{n_k}| > (\mu-\epsilon)^{n_k}.$$

Hence if $|z-a| > \frac{1}{\mu-\epsilon}$ the terms in the subseries

$$\sum_{k=1}^{\infty} a_{n_k}(z-a)^{n_k},$$

do not converge to 0 and hence the series must diverge.

Now suppose $0 = \mu = \limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |a_n|^{1/n}$. Then for $\epsilon > 0$, there is an N_0 such that for all $n \geq N_0$ one has

$$|a_n|^{1/n} < \epsilon\,.$$

and a K such that for all n $|a_n| < K\,\epsilon^n$. Thus

$$\left|\sum_{n=0}^{\infty} a_n(z-a)^n\right| \leq K \sum_{n=0}^{\infty} \epsilon^n\,|z-a|^n.$$

Since we may choose $\epsilon > 0$ such that $\epsilon\,|z-a| < 1$ for any given z we see that this series converges for all $z \in \mathbb{C}$.

The case where $\mu = \infty$ is omitted as it very similar to all other arguments used in this proof. □

Theorem 3.4. *Given a series $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$, f is analytic at each point within its radius of convergence. If R is the radius of convergence, then for $z \in D_R(a)$ the derivative is given by term by term differentiation,*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z-a)^{n-1},$$

and f' has radius of convergence R.

Proof. The last claim follows from the fact that $\lim_{n\to\infty} n^{1/n} = 1$ and therefore

$$\limsup |n a_n|^{1/n} = \limsup |a_n|^{1/n} = \frac{1}{R}.$$

The proof that the derivative is the desired series is identical to the calculation in real variables and hence we omit it. □

We have already shown that analytic functions are given by power series locally and that power series are differentiable within their radius of convergence. Thus we have a correspondence between the local definitions of analytic functions and their power series. This correspondence will help us prove the next theorem which states analytic functions which agree on a set with a limit point must be identical throughout their domains.

Theorem 3.5. *If f is analytic in a domain D and there exists a sequence $\{z_n\} \subseteq D$ with a limit point in D such that $f(z_n) = 0$ for all n, then $f \equiv 0$ on D.*

Proof. Let z_0 denote the limit point of z_n. By continuity we must have that $f(z_0) = 0$. By Theorem 3.1 there is an R such that for all $z \in D_R(z_0)$ f has a power series representation,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n.$$

Assuming that f is not identically 0 throughout $D_R(z_0)$ there exists a smallest integer m such that $a_m \neq 0$ and so we may write

$$f(z) = a_m(z-z_0)^m(1 + g(z-z_0)),$$

with $g(z-z_0) \to 0$ as $z \to z_0$. Taking z_n such that $z_n \in D_R(z_0)$ we see that

$$0 = f(z_n) = a_m(z-z_n)^m(1+g(z-z_n)) \neq 0,$$

a contradiction. Thus we have shown for any $z \in D$ there is an open neighborhood of z such that $f \equiv 0$ on this neighborhood. Since D is connected this implies $f \equiv 0$ on all of D. □

Proposition 3.6. *Suppose f is entire and $|f(z)| \leq M|z|^N$ for some fixed N and all $z \in \mathbb{C}$. Then f is a polynomial of degree at most N.*

Proof. f has a Taylor expansion about 0

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Now for C some circled centered at 0 of radius R, we have that for $n > N$

$$|a_n| = \left| \frac{1}{2\pi i} \int_C \frac{f(w)}{w^{n+1}} dw \right| \leq \frac{MR^{N+1}}{2\pi R^{n+1}},$$

and the right hand-sider goes to 0 as $R \to \infty$. Hence f is a polynomial of degree at most N. □

3.2. **Singularities and Laurent Series.** We now move on to the more general discussion of *Laurent Series*. We say a *singular point or singularity* of the function f is a point $z_0 \in \mathbb{C}$ such that f is not analytic at z_0. While we understand that away from singular points, functions have neat expressions in terms of power series, we wish to describe the behavior of a function near its singular points. This behavior is captured in the following theorem.

Theorem 3.7. *Suppose f is analytic on for all $z \in \mathbb{C}$ such that $r \leq |z-a| \leq R$ for some fixed $R, r > 0$ and $a \in \mathbb{C}$. Then there exists $\{a_n\}, \{b_n\}$ such that for z in this domain,*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}.$$

The series converge absolutely on $r \leq |z-a| \leq R$ and uniformly on $r < |z-a| < R$. Explictly,

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)^{n+1}} dw,$$
$$b_n = \frac{1}{2\pi i} \int_C f(w)(w-a)^{n-1} dw,$$

for any circle C centered at a between the circles C_R and C_r.

Proof. As f is analytic on $C_R(a)$ and $C_r(a)$ we may enlarge R slightly to R' and shrink r slightly to r' so that f is analytic on $r' \leq |z-a| \leq R'$. Then for fixed $z_0 \in \mathbb{C}$ such that $r \leq |z_0 - a| \leq R$, let γ be the contour given by a circle centered at z_0 such that $\gamma \subseteq \{z | r < |z-a| < R'\}$. By deformation of contours, we may then write,

$$\frac{1}{2\pi i} \int_{C_{R'}} \frac{f(w)}{w-z_0} dw = \frac{1}{2\pi i} \int_{C_{r'}} \frac{f(w)}{w-z_0} dw + \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z_0} dw.$$

By the Cauchy Integral Formula, this equation becomes,

$$f(z) = \frac{1}{2\pi i}\int_{C_{R'}} \frac{f(w)}{w-z_0}dw - \frac{1}{2\pi i}\int_{C_{r'}} \frac{f(w)}{w-z_0}dw.$$

Since,

$$\frac{1}{w-z_0} = \frac{1}{w-a} + \frac{z_0-a}{(w-a)^2} + \cdots + \frac{(z_0-a)^n}{(w-a)^{n+1}} + \frac{(z_0-a)^{n+1}}{(w-a)^{n+1}(w-z_0)},$$

we have that

$$\frac{1}{2\pi i}\int_{C_{R'}} \frac{f(w)}{w-z_0}dw = \frac{1}{2\pi i}\left(\int_{C_{R'}} \frac{f(w)}{w-a}dw + \cdots + \int_{C_{R'}} \frac{f(w)(z_0-a)^n}{(w-a)^{n+1}}dw\right) + R_n,$$

where

$$R_n = \frac{1}{2\pi i}\int_{C_{R'}} \frac{(z_0-a)^{n+1}}{(w-a)^{n+1}(w-z_0)} f(w)dw.$$

If we can show that $R_n \to 0$ as $n \to \infty$, we will have shown the computation of the a_n terms in the statement of the theorem. To this end, we note that,

$$|R_n| \leq \frac{1}{2\pi}\frac{|z_0-a|^{n+1}}{R'^{n+1}} 2\pi R' \max_{z\in C_{R'}} \frac{|f(w)|}{|w-z_0|}.$$

Since $C_{R'}$ is a compact set, $|f(w)|$ achieves a maximum, say M on the circle and as $|w-z_0| = |w-a-(z_0-a)| \geq ||w-a|-|z_0-a|| = |R'-|z_0-a||$, we obtain

$$|R_n| \leq \frac{M}{|R'-|z_0-a||}\left(\frac{|z_0-a|}{R'}\right)^n.$$

As the right hand side tends to 0 as $n \to \infty$ we get $R_n \to 0$ as $n \to \infty$ as desired. A very similar calculation for $-\frac{1}{2\pi i}\int_{C_{r'}} \frac{f(w)}{w-z_0}dw$ obtains the full result. □

With this theorem we can now classify various singularities for a function f. Suppose f has an isolated singularity at the point a and thus is analytic on $0 < |z-a| < R$ for some R. With a_n, b_n as in Theorem 3.5 we note if all $b_n = 0$ then f is in fact analytic at a. Such singularities are *removable*. If only finitely many $b_n \neq 0$, then f is said to have a pole at a of order corresponding to the largest n such that $b_n \neq 0$. Finally if infinitely many $b_n \neq 0$ then f is said to have an *essential singularity* at a. The next propositions will classify the behavior of f around the various types of singularities.

Theorem 3.8. *Suppose f has an isolated singularity at some $a \in \mathbb{C}$. If f is bounded in a neighborhood of a then a is a removable singularity.*

Proof. By Theorem 3.5 the for each $n \in \mathbb{N}$, the coefficient in the Laurent expansion b_n is given by

$$b_n = \frac{1}{2\pi i}\int_{C_r} f(w)(w-a)^{n-1}dw,$$

for $r < R$. Choosing r small enough so that C_r is contained in a neighborhood on which $|f|$ is bounded by M, we then have that

$$|b_n| \leq \frac{1}{2\pi} 2\pi r M r^{n-1}.$$

As r can be made arbitarily small, this implies $b_n = 0$ for all n. □

Now we consider the case where f has a pole of order m at a. Then we may write,

$$f(z) = \frac{b_m}{(z-a)^m} + \cdots + \frac{b_1}{z-a} + \sum_{n=0}^{\infty} a_n (z-a)^n,$$

and so

$$(z-a)^m f(z) = b_m + \cdots + b_1 (z-a)^{m-1} + \sum_{n=0}^{\infty} a_n (z-a)^{n+m}.$$

Letting $g(z) := (z-a)^m f(z)$ we see that g is analytic near a and $g(a) \neq 0$, and hence

$$\frac{1}{f(z)} = \frac{(z-a)^m}{g(z)},$$

is analytic near a and has a zero of order m. Whereas functions which are bounded near a singularity must have a removable singularity, we now show that functions which tend to infinity towards a singularity must have poles at those singularities.

Proposition 3.9. *Suppose f has an isolated singularity at some $a \in \mathbb{C}$. Then f has a pole at a if and only if*

$$\lim_{z \to a} |f(z)| = \infty.$$

Proof. Suppose first that a is a pole so that near a we may write f as

$$f(z) = \frac{g(z)}{(z-a)^m},$$

where m is the order of the pole, with g analytic near a and $g(a) \neq 0$. In a neighborhood, we then have $|g(z)| > M$ for some $M > 0$,

$$|f(z)| = \frac{|g(z)|}{|z-a|^m} \geq \frac{M}{|z-a|^m}.$$

As $\frac{M}{|z-a|^m} \to \infty$ as $z \to a$, we have the desired result.

Conversely suppose that $\lim_{z \to a} |f(z)| = \infty$. Then for fixed $M > 0$, we may find r such that $|z-a| \leq r$ implies $|f(z)| \geq M$. We then have

$$\frac{1}{f(z)} \leq \frac{1}{M},$$

which by Theorem 3.8 implies $\frac{1}{f(z)}$ is analytic at a. Setting $h(z) := \frac{1}{f(z)}$ we must have that $h(a) = 0$. Thus a is an isolated zero of f and so we may write $h(z) = (z-a)^m g(z)$ for some analytic g such that $g(a) \neq 0$. Thus f has a pole of order m at a. □

The behavior of f at an essential singularity is much more fascinating. We now prove what is known as the Casorati-Weierstraß theorem, although remark that much more is known—see Picard's Great Theorem.

Theorem 3.10 (Casorati-Weierstraß)**.** *Suppose f has an isolated essential singularity at $z_0 \in \mathbb{C}$. Then for any neighborhood N of z_0, $f(N)$ is dense in $\mathbb{C}$.*

Proof. We prove this by contradiction and hence suppose that for some neighborhood N about z_0, there exists a $c \in \mathbb{C}$ and an $\epsilon > 0$ such that $|f(w) - c| > \epsilon$ for all $w \in N$. The function $g(z) := \frac{1}{f(z)-c}$ is then analytic and bounded in N, and so has a removable singularity at z_0. Redefining g so it is analytic at z_0, $g(z_0) \neq 0$

by continuity and there is a $\delta > 0$ such that for $B_\delta(z_0) \subseteq N$, g is non-zero on this neighborhood. Expanding a power series for g at z_0 we get

$$g(z) = (z - z_0)^m h(z),$$

where $h(z_0) \neq 0$ and h is analytic near z_0. It follows that

$$f(z) - c = \frac{1}{(z - z_0)^m h(z)}.$$

Since $h(z_0) \neq 0$, $\frac{1}{h}$ is analytic near z_0 and so the above implies $f(z)$ has a pole of order m at z_0, a contradiction. □

Lemma 3.11 (Schwarz)**.** *Suppose f is analytic for $|z| < R$, $f(0) = 0$ and $|f(z)| \leq M$ for all $|z| < R$. Then,*

$$|f(z)| \leq \frac{M}{R}|z|,$$

for all $|z| < R$. Moreover if equality holds at any $z \neq 0$, then $f(z) = \lambda z$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = \frac{M}{R}$.

Proof. Set $g(z) := \frac{f(z)}{z}$. Then g is analytic on $|z| < R$ since its only potential pole is at $z = 0$ and g is bounded in a neighborhood of 0. In particular since $g(z) = \frac{f(z)-0}{z-0}$, the proper value to assign g at 0 to make g analytic is $g(0) = f'(0)$.

Now, let $R > \epsilon > 0$ and consider $C_\epsilon := \{z : |z| = R - \epsilon\}$. On C_ϵ we have

$$|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{M}{R - \epsilon}.$$

Taking $\epsilon \to 0$ we have the desired inequality. If equality holds at some $z \neq 0$, we have that $|g|$ attains a maximum in an open set and so by the maximum modulus principle g is constant. □

Theorem 3.12. *Let f be analytic on $\hat{\mathbb{C}}$ except for at isolated poles. Then f is a rational function.*

Proof. Since $\hat{\mathbb{C}}$ is compact, necessarily there are only finitely many poles. Otherwise compactness would imply the existence of an accumulation point amongst the poles, contradicting the assumption that they were isolated. Let $z_1, \ldots, z_n$ be the finite poles and allow for the possibility that ∞ is also a pole. Let

$$\sum_{\ell=1}^{m_j} \frac{A_\ell^{(j)}}{(z - z_j)^\ell},$$

be the principal part of f at the pole z_j, and

$$\sum_{\ell=1}^{m} A_\ell z^\ell,$$

be the principal part of the pole at ∞. Set

$$g(z) = f(z) - \sum_{j=1}^{n} \sum_{\ell=1}^{m_j} \frac{A_\ell^{(j)}}{(z - z_j)^\ell} - \sum_{\ell=1}^{m} A_\ell z^\ell.$$

$g(z)$ is by construction analytic on all of $\widehat{\mathbb{C}}$, and bounded since $\hat{\mathbb{C}}$ is compact. Thus by Liouville's Theorem, g is constant. From this it follows that f is a rational function. □

4. Möbius Transformations

Let $G := \mathrm{GL}_2(\mathbb{C})$. There is a well-defined action of G on $\mathbb{C}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az+b}{cz+d}$$

For a fixed element $g \in G$, the map $T_g(z) := g.z$ is said to be a *Möbius Transformation*. This action is not faithful, but upon passing to $\mathrm{PGL}_2(\mathbb{C}) := G/\mathbb{C}^\times$, it is faithful and so the group of Möbius transformations can be thought of abstractly as $\mathrm{PGL}_2(\mathbb{C})$. Often times we will simply think of such a transformation as an element of G which should be really be thought of an equivalence class of elements of G up to scalar multiplication. We will think about Möbius transformations as maps on the Riemann Sphere, $\hat{C}$—which is the one-point compactification of $\mathbb{C}$—by defining

$$\frac{a\infty + b}{c\infty + d} = \frac{a}{c},$$

provided $c \neq 0$ and otherwise setting this value to be ∞. In this way Möbius transformations define automorphisms of the Riemann Sphere.

Möbius transformations have the nice property of permuting circles on the Riemann Sphere. Under stereographic projection the circles on the unit sphere are in correspondence with the ordinary circles in $\mathbb{C}$, together with all straight lines. If two points $P, Q \in \mathbb{C}$ are related by reflection about the line ℓ in $\mathbb{C}$, they are said to be inverse to one another with respect to the *circle* ℓ—for the remainder of this section we will include straight lines in the discussion of circles as we always think of Möbius transformations as maps on the Riemann sphere. Given an ordinary circle $C = \{z \in \mathbb{C} : |z-a| = r\}$ in $\mathbb{C}$, P, Q are said to be inverse to C if P, Q, a are all colinear and $|P-a||Q-a| = r^2$. We now state but do not prove the theorem that Möbius transformations permute circles and preserve their inverse points.

Theorem 4.1. *Let T be a Möbius transformation, C a circle on the Riemann sphere, P, Q points inverse to C. Then $T(C)$ is a circle on the Riemann Sphere and $T(P), T(Q)$ are inverse to $T(C)$.*

Now, let z_1, z_2, z_3 be distinct points in $\hat{\mathbb{C}}$ and similarly for w_1, w_2, w_3. One can ask the natural question, is there a Möbius transformation T such that $Tz_i = w_i$ for each $i = 1, 2, 3$. To investigate this question, let z^* and w^* be finite points distinct for z_1, z_2, z_3 and w_1, w_2, w_3 respectively. Upon setting

$$U(z) := \frac{1}{z - z^*}, \quad V(w) := \frac{1}{w - w^*},$$

we replace z_i by $U(z_i)$ and w_i by $V(w_i)$ for each $i = 1, 2, 3$ so that we may assume all points are finite. We may do this, since if we find S such that $S(U(z_i)) = V(w_i)$ for each i, the desired transformation will be $V^{-1}SU$. Furthermore we claim we only need to find a $W, \widetilde{W}$ such that,

$$W(z_1) = 1, \quad W(z_2) = 0, \quad W(z_3) = \infty,$$
$$\widetilde{W}(w_1) = 1, \quad \widetilde{W}(w_2) = 0, \quad \widetilde{W}(w_3) = \infty,$$

since then $W\widetilde{W}^{-1}$ will be the desired map. To this end, define

$$W(z) := \lambda \frac{z - z_2}{z - z_3}, \quad \widetilde{W}(w) := \rho \frac{w - w_2}{w - w_3},$$

where λ, ρ scale $W, \widetilde{W}$ so that $W(z_1) = 1$ and $\widetilde{W}(w_1) = 1$ respectively. We have therefore that for any two pairs of distinct triples of points on the Riemann sphere, there is Möbius transformation mapping one triple to the other. To see this transformation is necessarily unique we must investigate fixed points of Möbius transformations.

Lemma 4.2. *A Möbius transformation T has at most 2 fixed points unless $T = \mathrm{Id}$.*

Proof. The condition that

$$\frac{az+b}{cz+d} = z,$$

is equivalent to saying z is a solution of the equation $az + b = cz^2 + dz$. Provided $c, b \neq 0$ and $a \neq d$ this is a polynomial of at most degree 2 and hence has at most 2 fixed points. In the case where $a = d$ and $c = b = 0$, then T is in the class of Id in $\mathrm{PGL}_2(\mathbb{C})$ and so acts as the identity. □

From this lemma we deduce that there is a *unique* transformation sending the distinct points $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ to distinct $w_1, w_2, w_3 \in \hat{\mathbb{C}}$ as if both T, S satisfied $Tz_i = w_i, Sz_i = w_i$ for all i, we'd have that z_1, z_2, z_3 are three distinct fixed points of $S^{-1}T$ and so $S^{-1}T = \mathrm{Id}$. Hence $S = T$. We remark now the effect of conjugation on fixed points. Simply note if T fixes z_0 and U is any other Möbius transformation then UTU^{-1} fixes $U(z_0)$. While a seemingly inconsipicuous remark, it will be crucial to simplify observations later on.

We now classify Möbius transformation and describe their beahvior. If T has a single fixed point z_0 we say T is *parabolic*. By the remark on conjugation we may conjugate T by some U so that UTU^{-1} fixes the identity. Such maps are of the form $z \mapsto z + b$ for some $b \in \mathbb{C}$. After performing some algbebra, one finds that all parabolics are of the form,

$$Tz = (1 + bz_0)z - z_0^2 b,$$

where z_0 is the unique fixed point.

If T has two unique fixed points, again may conjugate T by some U so that UTU^{-1} fixes both 0 and ∞. It follows that UTU^{-1} is of the form,

$$UTU^{-1}z = az,$$

for some $a \in \mathbb{C}$. If $|a| = 1$, then T is *elliptic* If $a \in \mathbb{R}$ but $|a| \neq 1$, then T is *hyperbolic*. Otherwise T is said to be *loxodromic*.

To further investigate properties of Möbius transformations we define the cross ratio of four points $z_1, \ldots, z_4 \in \mathbb{C}$ as

$$(z_1, z_2, z_3, z_4) := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

Upon fixing $z_2, z_3, z_4 \in \mathbb{C}$, the map $F(z) := (z, z_2, z_3, z_4)$ is a Möbius transformation. Note that $F(z_2) = 1, F(z_3) = 0, F(z_4) = \infty$ and by previous remarks we have that F is the unique transformation with this property.

Proposition 4.3. *For $z_1, \ldots, z_4, w_1, \ldots, w_4 \in \mathbb{C}$,*

$$(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4),$$

if and only if there exists a Möbius transformation T such that $Tz_i = w_i$ for $i = 1, 2, 3, 4$.

Proof. For one direction we prove more generally that for any S one has

$$(z_1, z_2, z_3, z_4) = (Sz_1, Sz_2, Sz_3, Sz_4).$$

To this end define $Tz := (z, Sz_2, Sz_3, Sz_4)$ and $Uz := (z, z_2, z_3, z_4)$. Then US^{-1} has the property that $Sz_2 \mapsto 1, Sz_3 \mapsto 0, Sz_4 \mapsto \infty$ and hence $US^{-1} = S$. Upon plugging in Sz_1 we get

$$(Sz_1, Sz_2, Sz_3, Sz_4) = T(Sz_1) = US^{-1}(Sz_1) = (z_1, z_2, z_3, z_4),$$

as desired.

For the converse direction, suppose that $(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4)$ and set $Tz := (z, z_2, z_3, z_4), Wz := (z, w_2, w_3, w_4)$. Then $W^{-1}T$ maps $z_2 \mapsto w_2, z_3 \mapsto w_3, z_4 \mapsto w_4$. Moreover,

$$W^{-1}Tz_1 = W^{-1}(z_1, z_2, z_3, z_4) = W^{-1}(w_1, w_3, w_3, w_4) = w_1.$$

Hence $W^{-1}T$ is the desired Möbius transformation between the z_i and w_i. □

Proposition 4.4. *Four points $z_1, \dots, z_4 \in \hat{\mathbb{C}}$ all lie on a unique circle in $\hat{\mathbb{C}}$ if and only if $(z_1, z_2, z_3, z_4) \in \mathbb{R}$.*

Proof. If all $z_i \in \mathbb{C}$ lie on some circle C in $\mathbb{C}$ upon applying $Tz : (z, z_2, z_3, z_4)$, by Theorem 4.1 we have that Tz_i all lie on the circle corresponding to the real axis in $\mathbb{C}$. Hence $Tz_1 = (z_1, z_2, z_3, z_4) \in \mathbb{R}$.

Conversely if $(z_1, z_2, z_3, z_4) \in \mathbb{R}$, let C denote the unique circle through z_2, z_3, z_4. For $Tz := (z, z_2, z_3, z_4)$ we have that $T(C) = \mathbb{R}$ and since $Tz_1 \in \mathbb{R}$ it follows that z_1 also lies on the circle C. □

Proposition 4.5. *The two points z_0, z_1 are inverse to the circle C if and only if for any $z_2, z_3, z_4 \in C$,*

$$(z_0, z_2, z_3, z_4) = \overline{(z_1, z_2, z_3, z_4)}.$$

Proof. If z_0, z_1 are inverse to C, upon applying $Tz := (z, z_2, z_3, z_4)$ for any $z_2, z_3, z_4 \in C$, we have that $T(C) = \mathbb{R}$ and Tz_0, Tz_1 are inverse to $\mathbb{R}$. In other words , $Tz_0 = \overline{Tz_1}$ as desired.

Conversely let C be a circle such that for any $z_2, z_3, z_4 \in C$ we have

$$(z_0, z_2, z_3, z_4) = \overline{(z_1, z_2, z_3, z_4)}.$$

Setting $Tz := (z, z_2, z_3, z_4)$ we have that $T(C) = \mathbb{R}$ and Tz_0, Tz_1 are inverse to $T(C)$. Applying T^{-1} obtains the result. □

5. The Residue Theorem, Examples, and Consequences

5.1. **The Theorem and Many Examples.** We now turn to the discussion of residues. If f is analytic in $0 < |z - a| < R$ for some $a \in \mathbb{C}$, we can write f as a Laurent Series,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}.$$

The value b_1 in this expansion is the *residue for f at a*. We single it out, and give it this name since if C is a simple closed contour enclosing a and no other singularities of f, we have that

$$\frac{1}{2\pi i} \int_C f dz = b_1,$$

since the only term that does not vanish in the Laurent Series upon taking the integral is the first term of the principal part.

Theorem 5.1 (Cauchy's Residue Theorem)**.** *Let f be analytic inside and on a simple closed contour, except at the singularities $a_1, \ldots, a_n$. Then*

$$\int_C f dz = 2\pi i \sum_{i=1}^{n} \text{Res}(f; a_i).$$

Proof. Letting C_i denote a simply closed contour around a_i which only encloses the singularity a_i, by general deformation of contours we have

$$\int_C f dz = \sum_{i=1}^{n} \int_{C_i} f dz = 2\pi i \sum_{i=1}^{n} \text{Res}(f; a_i).$$

□

We now do many examples to demonstrate the utility of this theorem, despite the innoculous nature of its proof. Consider the real integral $\int_0^\infty \frac{\sin x}{x} dx$. Since the integrand is even, it suffices to compute $\int_{-\infty}^\infty \frac{\sin x}{x} dx$. We will compute this integral by considering the contour integral of $\frac{e^{iz}}{z}$ along the contour $C_{R,\rho}$ is composed of the the half circle of radius R oriented counter clockwise, denoted Γ_R, followed by the line segment from $-R$ to $-\rho$, then the half circle of radius ρ oriented clockwise, denoted Γ_ρ and finally followed by the line segment from ρ to R. Since $\frac{e^{iz}}{z}$ inside and on $C_{R,\rho}$ we have that

$$\int_{C_{R,\rho}} \frac{e^{iz}}{z} dz = 0,$$

and splitting up the integral we get

$$\int_{-R}^{-\rho} \frac{e^{ix}}{x} dx + \int_{\rho}^{R} \frac{e^{ix}}{x} dx + \int_{\Gamma_R} \frac{e^{iz}}{z} dz + \int_{\Gamma_\rho} \frac{e^{iz}}{z} dz = 0. \tag{1}$$

For the first two terms, using a change of variables from x to $-x$ in the first term we get

$$\int_{-R}^{-\rho} \frac{e^{ix}}{x} dx + \int_{\rho}^{R} \frac{e^{ix}}{x} dx = \int_{\rho}^{R} \frac{e^{ix} - e^{-ix}}{x} dx = 2i \int_{\rho}^{R} \frac{\sin x}{x} dx.$$

Hence we need only compute the other terms to evaluate our desired integral. For Γ_ρ we split the integral as

$$\int_{\Gamma_\rho} \frac{e^{iz}}{z} dz = \int_{\Gamma_\rho} \frac{e^{iz} - 1}{z} dz + \int_{\Gamma_\rho} \frac{1}{z} dz.$$

$\int_{\Gamma_\rho} \frac{1}{z} dz$ is simply computed to be $-\pi i$—negative since we orient Γ_ρ in the clockwise direction. Moreover $\frac{e^{iz}-1}{z}$ is entire and hence has a primitive say F, so that

$$\int_{\Gamma_\rho} \frac{e^{iz} - 1}{z} dz = F(-\rho) - F(\rho).$$

Letting $\rho \to 0$ we see that

$$\lim_{\rho \to 0} \int_{\Gamma_\rho} \frac{e^{iz} - 1}{z} dz = 0.$$

Hence altogether we have that

$$\lim_{\rho\to 0}\int_{\Gamma_\rho}\frac{e^{iz}}{z}dz = -\pi i.$$

To conclude we claim $\int_{\Gamma_R}\frac{e^{iz}}{z}dz \to 0$ as $R\to\infty$. To this end we parametrize the integral via $z(\theta) = Re^{i\theta}$ to get that

$$\left|\int_{\Gamma_R}\frac{e^{iz}}{z}dz\right| = \left|i\int_0^\pi e^{iRe^{i\theta}}d\theta\right| \le \int_0^\pi |e^{iRe^{i\theta}}|d\theta = \int_0^\pi e^{-R\sin\theta}d\theta.$$

To estimate the right most integral, let $\epsilon > 0$ be arbitrary and split the integral as

$$\int_0^\pi e^{-R\sin\theta}d\theta = \int_0^\epsilon e^{-R\sin\theta}d\theta + \int_\epsilon^{\pi-\epsilon} e^{-R\sin\theta}d\theta + \int_{\pi-\epsilon}^\pi e^{-R\sin\theta}d\theta.$$

For the outer integrals, $e^{-R\sin\theta} \le 1$ within their domains of integration and so,

$$\int_0^\epsilon e^{-R\sin\theta}d\theta + \int_{\pi-\epsilon}^\pi e^{-R\sin\theta}d\theta \le \int_0^\epsilon 1d\theta + \int_{\pi-\epsilon}^\pi 1d\theta = 2\,\epsilon\,.$$

Moreover on $[\epsilon, \pi-\epsilon]$ there exists a $\delta > 0$ so that $\sin\theta > \delta$, giving

$$\int_\epsilon^{\pi-\epsilon} e^{-R\sin\theta}d\theta \le \int_0^\pi e^{-R\delta}d\theta = \pi e^{-R\delta}.$$

Hence for arbitrary $\epsilon > 0$, we get that

$$\left|\int_{\Gamma_R}\frac{e^{iz}}{z}dz\right| \le 2\,\epsilon + \pi e^{-R\delta},$$

and so

$$\left|\int_{\Gamma_R}\frac{e^{iz}}{z}dz\right| \le \pi e^{-R\delta}.$$

Taking $R\to\infty$ we get that $\int_{\Gamma_R}\frac{e^{iz}}{z}dz \to 0$. Taking $R\to\infty, \rho\to 0$ in Equation 3 we therefore obtain,

$$2i\int_0^\infty \frac{\sin x}{x}dx - \pi i = 0,$$

giving $\int_0^\infty \frac{\sin x}{x}dx = \pi/2$.

Thus as we have demonstrated the residue theorem allows us to calculate real-valued integrals that are not able to be calculated using ordinary techniques learned in calculus. Before proceeding with more examples, we present a short lemma which is useful for calculating residues at poles.

Lemma 5.2. *Say f has an isolated pole of order m at $a \in \mathbb{C}$. Then*

$$\operatorname{Res}(f;a) = \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z).$$

Proof. Having a pole of order m at a means f has a Laurent Series expansion near a of the form,

$$f(z) = \frac{b_m}{(z-a)^m} + \cdots + \frac{b_1}{(z-a)} + a_0 + a_1(z-a) + \ldots$$

so

$$(z-a)^m f(z) = b_m + \cdots + b_1(z-a)^{m-1} + \ldots$$

Thus

$$\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z) = (m-1)!b_1 + g(z),$$

where $\lim_{z\to a} g(z) = 0$, giving the result. □

For our next example we consider the integral,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ixt}}{a^2+x^2}dx,$$

for some fixed $a, t > 0$. To calculate this integral, we consider the contour integral

$$\int_{C_R} \frac{e^{-izt}}{a^2+z^2}dz,$$

where C_R denotes the contour consisting of the half circle of radius R, denoted Γ_R together with the line segment joining $-R$ to R. $\frac{e^{-izt}}{a^2+z^2}$ has a single simple pole at $z = ai$ in this contour which we compute via the formula,

$$\text{Res}(f; ai) = \lim_{z\to ai}(z-ai)\frac{e^{-izt}}{a^2+z^2} = \lim_{z\to ai}\frac{e^{-izt}}{z+ai} = \frac{e^{-at}}{2ai}.$$

Hence by the residue theorem

$$\int_{\Gamma_R} \frac{e^{-izt}}{a^2+z^2}dz + \int_{-R}^{R} \frac{e^{-ixt}}{a^2+x^2}dx = \frac{\pi e^{-at}}{a}.$$

We claim the integral along Γ_R tends to 0 as $R \to \infty$. To see this parametrize Γ_R via $z = Re^{i\theta}$ to get

$$\left|\int_{\Gamma_R} \frac{e^{-izt}}{a^2+z^2}dz\right| = \left|\int_0^{\pi} \frac{e^{itRe^{i\theta}}}{a^2+R^2e^{i2\theta}}iRe^{i\theta}d\theta\right| \le \int_0^{\pi} \frac{1}{R}d\theta.$$

The right most integral tending to 0 as $R \to 0$ gives the desired result. Hence,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ixt}}{a^2+x^2}dx = \frac{e^{-at}}{2a}.$$

Now consider the integral,

$$\int_{-\infty}^{\infty} \frac{\cos(xt)}{a^2-x^2}dx,$$

where again $a, t, > 0$. Note that this integral has two singularities along the real axis, and in fact if one compute the standard Riemann integral near these singularities one will get that this integral diverges. If however we apply the theory of residues, there will be a notion for which this limit exists. To this end we consider the contour integral

$$\int_{C_{R,\rho}} \frac{e^{izt}}{a^2-z^2}dz,$$

where $C_{R,\rho}$ denotes the contour which consists of the half circle of radius R, oriented counter clockwise and denoted by Γ_R, followed by the line segment from $-R$ to $-a-\rho$, followed by the half circle of radius ρ centered at $-a$ oriented clockwise and denoted γ_1, followed by the line segment joining $-a+\rho$ to $a-\rho$, then the half circle of radius ρ centered at a oriented clockwise and denoted γ_2, and finally followed by the line segment connecting $a+\rho$ to R. The integrand is analytic on and inside this contour, giving this integral is 0. Splitting this integral and writing $f = \frac{e^{izt}}{a^2-z^2}$, we get

$$\int_{\Gamma_R} f dz + \int_{-R}^{-a-\rho} f dx + \int_{-a+\rho}^{a-\rho} f dx + \int_{a+\rho}^{R} f dx = \int_{-\gamma_1} f dz + \int_{-\gamma_2} f dz \tag{2}$$

For the integrals on the real line, note that

$$\int_{-R}^{-a-\rho} \frac{e^{ixt}}{a^2-x^2} dx = \int_{a+\rho}^{R} \frac{e^{-ixt}}{a^2-x^2} dx,$$
$$\int_{-a+\rho}^{0} \frac{e^{ixt}}{a^2-x^2} dx = \int_{0}^{a-\rho} \frac{e^{-ixt}}{a^2-x^2} dx$$

So that we may write,

$$\int_{-R}^{-a-\rho} f dx + \int_{-a+\rho}^{a-\rho} f dx + \int_{a+\rho}^{R} f dx = \int_{0}^{a-\rho} \frac{2\cos(xt)}{a^2-x^2} dx + \int_{a+\rho}^{R} \frac{2\cos(xt)}{a^2-x^2} dx.$$

For Γ_R simply note that,

$$\left| \int_{\Gamma_R} \frac{e^{itz}}{a^2-z^2} dz \right| \leq \frac{\pi R}{R^2-a^2},$$

and so $\int_{\Gamma_R} \frac{e^{itz}}{a^2-z^2} dz \to 0$ as $R \to \infty$. Hence we need only calculate the integrals around γ_1 and γ_2. Due to the minus signs showing up in Equation 2 we calculate the integrals around these curves oriented counter clockwise. To this end, we may write

$$\frac{e^{itz}}{a^2-z^2} = \frac{e^{itz}}{(a-z)(a+z)},$$

so that

$$(z-a)f = \frac{-e^{itz}}{z+a} = \frac{-e^{ita}}{2a} + g(z),$$

where $g(z)$ is analytic at a and $g(a) = 0$. Hence we may write

$$f = \frac{-e^{ita}}{2a(z-a)} + \frac{g(z)}{z-a}.$$

From this we see

$$\int_{\gamma_2} f dz = \int_{\gamma_2} \frac{-e^{ita}}{2a(z-a)} dz + \int_{\gamma_2} \frac{g(z)}{z-a} dz = \frac{-e^{ita}\pi i}{2a} + \int_{\gamma_2} \frac{g(z)}{z-a} dz.$$

Moreover since $\frac{g(z)}{z-a}$ is analytic in a neighborhood of a, it has a primitive and so as $\rho \to 0$ the endpoints of γ_2 tend towards one another giving

$$\lim_{\rho \to 0} \int_{\gamma_2} \frac{g(z)}{z-a} dz = 0.$$

We proceed similarly to evaluate γ_1. Note that

$$(z+a)f = \frac{e^{itz}}{a-z} = \frac{-e^{itz}}{z-a} = \frac{e^{-ita}}{2a} + h(z),$$

where h is analytic and $h(-a) = 0$. Hence the laurent expansion for f near $-a$ is

$$f = \frac{e^{-ita}}{2a(z+a)} + \frac{h(z)}{z+a}.$$

Integrating we therefore obtain,

$$\int_{\gamma_1} f dz = \int_{\gamma_1} \frac{e^{-ita}}{2a(z+a)} dz + \frac{h(z)}{z+a} = \frac{\pi i e^{-ita}}{2a} + \frac{h(z)}{z+a},$$

where again for the same reason, the second integral tends to 0 as $\rho \to 0$. Hence altogether we have shown,

$$\lim_{\rho \to 0} \int_{-\gamma_1} f dz + \int_{-\gamma_2} f dz = \frac{\pi i (e^{-ita} - e^{ita})}{2a} = \frac{\pi}{a} \sin(at).$$

Thus using Equation 2 we obtain that

$$2 \int_0^\infty \frac{\cos(xt)}{a^2 - x^2} = \int_{-\infty}^\infty \frac{\cos(xt)}{a^2 - x^2} = \frac{\pi}{a} \sin(at).$$

Note however that this indefinite integral is not the standard sum of independent limits near the singularities of the integrand. Instead this indefinite is calculated by integrating near the singularities of the integrand by taking a symmetric limit of the integrals up to a neighborhood of the singularity. Sometimes this is referred to as the *principal value* of the integral.

For our next example, we let $0 < a < 1$ and consider

$$\int_0^\infty \frac{x^{a-1}}{1+x} dx.$$

In order to evaluate this integral, we conside the contour $C_{R,\rho}$ which consts of the circle of radius R save for a strip of height ρ about the positive real axis and a portion of the cirlce of radius ρ. This contour is demonstrated in the figure below. We then aim to compute,

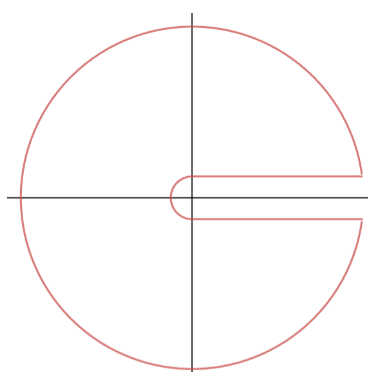


FIGURE 1. Contour $C_{R,\rho}$

$$\int_{C_{R,\rho}} \frac{z^{a-1}}{1+z} dz.$$

Inside and on this contour, the principal branch of the log is well-defined and so we can write

$$z^{a-1} = e^{(a-1)\log(z)}.$$

Being analytic, the only pole for our integrand is at $z = -1$. We calculate the residue,

$$\operatorname{Res}\left(\frac{z^{a-1}}{1+z};-1\right) = \lim_{z\to -1} z^{a-1} = \lim_{z\to -1} e^{(a-1)\log(z)} = e^{(a-i)\pi i}.$$

Thus via the residue theorem,

$$\int_{C_{R,\rho}} \frac{z^{a-1}}{1+z}dz = 2\pi i e^{(a-i)\pi i}.$$

On the other hand this integral is given by

$$\int_{C_{R,\rho}} \frac{z^{a-1}}{1+z}dz = \int_{\Gamma_R} \frac{z^{a-1}}{1+z}dz + \int_{\Gamma_\rho} \frac{z^{a-1}}{1+z}dz + \int_{\ell_1} \frac{z^{a-1}}{1+z}dz + \int_{\ell_2} \frac{z^{a-1}}{1+z}dz,$$

where Γ_R, Γ_ρ denote the obvious pieces of the circles and ℓ_1, ℓ_2 denote the two lines joining these circles. Note that,

$$\left|\int_{\Gamma_R} \frac{z^{a-1}}{1+z}dz\right| \leq \frac{2\pi R R^{a-1}}{R-1}.$$

The right hand side tends to 0 as $R\to\infty$ since $a < 1$ and hence so does the integral. Similarly,

$$\left|\int_{\Gamma_\rho} \frac{z^{a-1}}{1+z}dz\right| \leq \frac{2\pi\rho\rho^{a-1}}{1-\rho}.$$

Since the right hand side tends to 0 as $\rho \to 0$, we also get $\int_{\Gamma_\rho} \frac{z^{a-1}}{1+z}dz \to 0$. Hence

$$2\pi i e^{(a-1)\pi i} = \lim_{\substack{R\to\infty\\ \rho\to 0}} \int_{\ell_1} \frac{z^{a-1}}{1+z}dz + \int_{\ell_2} \frac{z^{a-1}}{1+z}dz.$$

It is important to note that while both ℓ_1 and ℓ_2 converge to the positive real axis with opposite orientations in this integral, these integrals do not cancel out, as ℓ_2 occurs after having rotated 2π radians about the origin and therefore changes the value of x^{a-1}. Instead this limit is given by

$$2\pi i e^{(a-i)\pi i} = \int_0^\infty \frac{x^{a-1}}{1+x}dx - \int_0^\infty \frac{e^{(a-1)2\pi i}x^{a-1}}{1+x}dx = (1 - e^{(a-1)2\pi i})\int_0^\infty \frac{x^{a-1}}{1+x}dx.$$

Hence

$$\int_0^\infty \frac{x^{a-1}}{1+x}dx = \frac{2\pi i e^{(a-1)\pi i}}{1-e^{(a-1)2\pi i}}.$$

After some algebraic manipulations it can be shown that this value is precisely $\frac{\pi}{\sin(a\pi)}$.

For our final example for the time being, we consider an integral along the a line parallel to the imaginary axis. Fixing $a, c > 0$ with $a < 1$, consider

$$\int_{c-i\infty}^{c+\infty} \frac{a^z}{z^2}dz.$$

To evaluate this integral we consider the contour which consists of the segment of this line which is connecting by the circle of radius R centered at the origin as demonstrated in the figure below. Inside this contour, the branch of log defined via

$$\log(z) = \log|z| + \operatorname{Arg}(z)i,$$

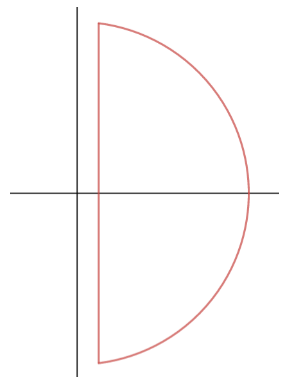


FIGURE 2. Contour C_R

for $-\pi < \mathrm{Arg}(z) < \pi$ is well-defined and analytic. Letting Γ_R denote the portion of the circle, on Γ_R we have $a^z = e^{z\log(a)}$. On Γ_R $z = Re^{i\theta}$ with $\mathrm{Re}(z) = R\cos\theta$. Thus we see on Γ_R,

$$|a^z| = |e^{z\log(a)}| = e^{\mathrm{Re}(z\log(a)} = e^{R\cos\theta\log a}.$$

Since $\log(a) < 0$, to maximize $e^{R\cos\theta\log a}$ we must minimize $R\cos\theta$. As constructed our contour gives rise to some M so that $R\cos\theta \geq M$ on all of Γ_R. Hence

$$|a^z| \leq e^{M\log a},$$

on Γ_R. Therefore,

$$\left|\int_{\Gamma_R} \frac{a^z}{z^2} dz\right| \leq \frac{\pi Re^{M\log(a)}}{R^2},$$

which tends to 0 as $R \to \infty$. Moreover since $\frac{a^z}{z^2}$ is analytic inside and on this contour, the whole integral evaluates to 0, giving

$$\int_{c-i\infty}^{c+\infty} \frac{a^z}{z^2} dz = 0,$$

when $a < 1$.

If on the other hand $a > 1$, the same contour will not suffice. Instead we join our portion of the line segment with the other piece of the circle of radius R as demonstrated in the figure below.

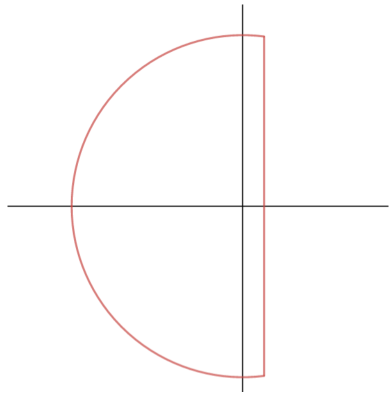


FIGURE 3. Contour C_R

To evaluate,

$$\int_{C_R} \frac{a^z}{z^2} dz$$

we note there is a single pole of order 2 at the origin inside this contour. This time we take the principal brance To find the residue simply observe,

$$\frac{a^z}{z^2} = \frac{1}{z^2} e^{z \log(a)} = \frac{1}{z^2}\left(1 + z\log(a) + \frac{z^2(\log(a))^2}{2}\right),$$

and so

$$\operatorname{Res}(f;0) = \log(a).$$

Hence by the resiude theorem,

$$\int_{C_R} \frac{a^z}{z^2} dz = 2\pi i \log(a).$$

Letting Γ_R denote the segment of the circle in the contour, we see on Γ_R for $z = x + iy$,

$$|a^z| = |e^{z\log(a)}| = e^{x\log(a)} \leq e^{c\log(a)}.$$

Thus

$$\left|\int_{\Gamma_R} \frac{a^z}{z^2} dz\right| \leq \frac{2\pi R e^{c\log(a)}}{R^2}$$

and so $\int_{\Gamma_R} \frac{a^z}{z^2} dz \to 0$ as $R \to \infty$. Hence

$$\int_{c-i\infty}^{c+\infty} \frac{a^z}{z^2} dz = 2\pi i \log(a),$$

when $a > 1$.

5.2. **The Argument Principle.** We now move on to discussing consequences of the residue theorem.

Theorem 5.3. *Let f be analytic on a simple closed contour C and analytic inside C except for at a a finite number of poles. Further supposing $f(z) \neq 0$ for any z on C, we have*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz = N - P,$$

where N denotes the number of zeroes of f inside C—counted with multiplicity—and P denotes the number of poles, again counted with multiplicity.

Proof. By the residue theorem we have that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \sum_{i=1}^{n} \operatorname{Res}\Big(\frac{f'}{f}; z_i\Big),$$

where $z_1, \ldots, z_n$ denote the poles of $\frac{f'}{f}$. The potential poles for $\frac{f'}{f}$ are precisely the zeroes of f and the poles of f'—which are in bijection with the poles of f. If f has a zero of order m at z_0 inside C the nearby z_0 we may write,

$$f(z) = (z - z_0)^m g(z),$$

where $g(z_0) \neq 0$ and is analytic. Then

$$\frac{f'}{f} = \frac{m(z-z_0)^{m-1}g(z) + (z-z_0)^m g'(z)}{(z-z_0)^m g(z)} = \frac{m}{z-z_0} + \frac{g'}{g}.$$

Since $\frac{g'}{g}$ is analytic, it follows that

$$\operatorname{Res}\Big(\frac{f'}{f}; z_0\Big) = m,$$

precisely the order of the zero.

Similarly if z_0 is now a pole of order k then nearby z_0, we may write

$$f(z) = \frac{g(z)}{(z-z_0)^k},$$

where $g(z_0) \neq 0$ and g analytic. Then

$$\frac{f'}{f} = \frac{\frac{(z-z_0)^k g'(z) - g(z)k(z-z_0)^{k-1}}{(z-z_0)^{2k}}}{\frac{g(z)}{(z-z_0)^k}} = \frac{g'}{g} - \frac{k}{z-z_0}.$$

Again as $\frac{g'}{g}$ is analytic we get that

$$\operatorname{Res}\Big(\frac{f'}{f}; z_0\Big) = -k.$$

Hence

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \sum_{i=1}^{n} \operatorname{Res}\Big(\frac{f'}{f}; z_i\Big) = N - P.$$

□

Say f is continuous on a simple closed contour C. Parametrizing C by $z(t)$ for $a \leq t \leq b$, we set

$$\Delta_C(\operatorname{Arg} f(z)) := \operatorname{Arg} f(z(b)) - \operatorname{Arg} f(z(a)),$$

to be the change in argument in f as $z(t)$ traces out C. For example if $f(z) = z^2$ and C is the unit circle parametrized via $z(\theta) = e^{i\theta}$ for $0 \leq \theta \leq 2\pi$, then

$$\Delta_C(\operatorname{Arg} f(z)) = 4\pi - 0 = 4\pi.$$

Theorem 5.4 (Argument Principle)**.** *Let f be analytic on a simple closed contour C and analytic inside C except for at a a finite number of poles. Further supposing $f(z) \neq 0$ for any z on C, we have*

$$\frac{1}{2\pi i}\Delta_C(\operatorname{Arg} f(z)) = N - P.$$

Proof. By theorem 5.3 we have that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P.$$

Moreover if $f(z) \neq 0$, there is a neighborhood of $f(z)$ where $\log(f(z))$ is well-defined and in this neighborhood,

$$\frac{f'(z)}{f(z)} = \frac{d}{dz}\log(f(z)).$$

If C is parametrized by $z(t)$ for $a \leq t \leq b$, we therefore have

$$\begin{aligned}\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz &= \frac{1}{2\pi i}\int_C \frac{d}{dz}\log(f(z))dz, \\ &= \frac{1}{2\pi i}[\log(f(z(b)) - \log(f(z(a))], \\ &= \frac{1}{2\pi i}\Delta_C(\operatorname{Arg} f(z)).\end{aligned}$$

□

One of the most important corollaries of the argument principle is what is known as Rouche's Theorem.

Theorem 5.5 (Rouche's Theorem)**.** *Suppose f, g are analytic inside and on a simple closed curve C and suppose that $|f| > |g|$ on C. Then f and $f + g$ have the same number of zeroes—with multiplicity—inside C.*

Proof. Note that via our assumptions, f and therefore $f + g$ are both non-zero on C. Hence by Theorem 5.3,

$$N_{f+g} = \frac{1}{2\pi i}\int_C \frac{(f+g)'}{(f+g)}dz = \frac{1}{2\pi i}\int_C \frac{f'}{f}dz + \frac{1}{2\pi i}\int_C \frac{(1+\frac{g}{f})'}{1+\frac{g}{f}}dz.$$

We know that $\frac{1}{2\pi i}\int_C \frac{f'}{f}dz = N_f$ and by the argument principle, $\frac{1}{2\pi i}\int_C \frac{(1+\frac{g}{f})'}{1+\frac{g}{f}}dz$ is the winding number of the curve $(1 + \frac{g}{f})(C)$. Note that this number is necessarily 0 as since $|g/f| < 1$, $(1 + \frac{g}{f})(C)$ is entirely contained in the circle of radius 1 about 1 in $\mathbb{C}$. Thus $(1 + \frac{g}{f})(C)$ cannot possibly wrap around the origin and hence its winding number is 0. This gives

$$\frac{1}{2\pi i}\int_C \frac{(1+\frac{g}{f})'}{1+\frac{g}{f}}dz = 0$$

and so

$$N_{f+g} = N_f.$$

□

5.3. **Consequences of Rouche's Theorem.** One of the consequences of Rouche's Theorem is a very simple proof of the Fundamental Theorem of Algebra. If $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, we can set $f(z) = a_n z^n$ and $g(z) = a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$. Then since f has higher degree than g,

$$\lim_{z\to\infty} |\frac{g}{f}| \to 0.$$

Thus we can find an $R > 0$ such that for $|z| \geq R$ one has $|g/f| < 1/2$. On the circle of radius R, then $|f| > |g|$ and so by Rouche's theorem f and $f + g$ have the same number of zeroes inside D_R. $f + g = p$ and f clearly has n zeroes inside D_R from which we conclude that p has n zeroes inside D_R. Another important consequence of Rouche's Theorem is what's known as Hurwitz's Theorem.

Theorem 5.6 (Hurtwitz's Theorem)**.** *Let $\{f_n\}_1^\infty$ be a sequence of analytic functions on a domain D. Suppose $f_n \to f$ uniformly on every compact subset of D and $f \not\equiv 0$. Then $z_0 \in D$ is a zero of order m for f if and only if for any $\epsilon > 0$ small enough and any $N \in \mathbb{N}$ large enough, each f_n for $n \geq N$ has precisely m zeroes—counted with multiplicity—inside $D_\epsilon(z_0)$.*

Proof. Let us first recall the argument that f must be analytic. We show this using Morera's Theorem. To this end, let C be a simple closed curve in D. C is a compact set and so $f_n \to f$ uniformly on C. That is, for any $\epsilon > 0$ there exists an N such that for all $n \geq N$ and all $z \in \mathbb{C}$, $|f_n(z) - f(z)| \leq \epsilon$. Thus for $n \geq N$,

$$\left|\int_C f_n dz - \int_C f dz\right| = \left|\int_C f_n - f dz\right| \leq \int_C |f_n - f| dz \leq \epsilon L,$$

where L denotes the length of C. Taking $\epsilon \to 0$ we see that $\int_C f_n \to \int_C f$. But as each f_n are analytic $\int_C f_n = 0$ for all n and therefore $\int_C f dz = 0$ and so by Morera's Theorem, f must be analytic.

Now since $f \not\equiv 0$, it has isolated zeroes, so for $z_0 \in D$ a zero of order m for f, we let r be small enough so that z_0 is the only zero for f inside of $D_r(z_0)$. Necessarily $|f(z)| > 0$ for all $|z - z_0| = r$, and moreover since the circle $C_r(z_0)$ is compact there is a minimum achieved, say

$$\delta := \min_{|z-z_0|=r} |f(z)| > 0.$$

As $f_n \to f$ uniformly on $|z - z_0| = r$ we may find an N such that for all $n \geq N$ and all z with $|z - z_0| = r$,

$$|f_n(z) - f(z)| \leq \frac{\delta}{2}.$$

Thus for $n \geq N$, $|f(z)| > |f_n(z) - f(z)|$ on $C_r(z_0)$ and so by Rouche's Theorem, f and f_n have the same number of zeroes inside $D_r(z_0)$. By construction f has precisely m such zeroes and so the same holds true for f_n. □

We can now combine the previous two results to obtain the following.

Theorem 5.7. *Let f be analytic in a domain D with Taylor expansion,*

$$f(z) = a_0 + a_N (z - a)^N + \ldots$$

where $a_N \neq 0$ and $N \geq 1$. Then for $r > 0$ sufficiently small, there is a $\rho(r) > 0$ such that f takes every value in $\{|w - a_0| < \rho\}$ exaclty N times in $|z - a| < r$.

Proof. Note that,

$$\lim_{z\to a}\frac{f(z)-a_0}{(z-a)^N}=a_N\neq 0.$$

Hence there is a $\kappa>0$ such that for $|z-a|\leq\kappa$ one has

$$\left|\frac{f(z)-a_0}{(z-a)^N}\right|>\frac{1}{2}|a_N|.$$

Hence for $|z-a|\leq\kappa$,

$$|f(z)-a_0|>\frac{1}{2}|z-a|^N|a_N|>0,$$

and hence $f(z)$ only takes the value a_0 in $|z-a|\leq\kappa$ at $z=a$ and by an application of Rouche's Theorem we see that f takes the value a_0 with multiplicity N.

Now for $0<r\leq\kappa$ let $\rho(r):=\frac{1}{2}|a_N|r^N$. Fix a value c such that $|c-a_0|<\rho(r)$. We aim to that f takes the value c precisely N times in $|z-a|<r$. Note that for $c=a_0$ we have already justified that f takes the value a_0 with multiplicity N inside $|z-a|<r$. If $c\neq a_0$, we aim to apply Rouche's Theorem to the functions $f(z)-a_0$ and $-(c-a_0)$. Note on $|z-a|=r$,

$$\left|\frac{f(z)-a_0}{(z-a)^N}\right|=\frac{|f(z)-a_0|}{r^N}>\frac{1}{2}|a_N|,$$

where the inequality follows from the fact that $r\leq\kappa$. This gives,

$$|f(z)-a_0|>\frac{1}{2}|a_N|r^N=\rho(r)>|c-a_0|.$$

Applying Rouche's Theorem, $f(z)-a_0$ and $f(z)-a_0-(c-a_0)=f(z)-c$ have the same number of zeroes inside $|z-a|=r$. Having already justified that $f(z)-a_0$ has N zeroes with multiplicity inside this circle we conclude the same for $f(z)-c$, proving the result. □

A corollary of the results we have just shown is the Inverse Function Theorem.

Corollary 5.8 (Inverse Function Theorem)**.** *Suppose f is analytic on a domain D, $f'(a)\neq 0$ for some $a\in D$. Then f is locally one to one near z_0 and has analytic inverse at a. Moreover if $g=f^{-1}$ and $w_0=f(z_0)$, then*

$$g'(w_0)=\frac{1}{f'(z_0)}.$$

Proof. Write $f(a)=a_0$. The condition that $f'(z_0)\neq 0$ says that the coefficient a_1 in the Taylor Expansion of f at z_0 is non-zero. That is, we may apply Theorem 5.7 in the case where $N=1$. Thus there exists an $r>0$ such that for some $\rho(r)$ every value in $|w-a_0|<\rho(r)$ is taken exactly once in $|z-a|<r$. Set $U:=\{|w-a_0|<\rho\}$. Since $f'(a)\neq 0$, $f'(z)\neq 0$ for z near a by continuity and so we simply remark that we may choose r above so that $f'(z)\neq 0$ on $|z-a|<r$. By continuity we have that U is open and so $f^{-1}(U)$ is open and thus define V to be the open set $f^{-1}(U)\cap\{|z-a|<r\}$. We claim that V is necessarily a neighborhood of a. Clearly any point in V cannot map into $\{|w-a_0|>\rho(r)\}$ by continuity and the max-modulus principle guarantees that since V is an open set, no such point can map onto $|w-a_0|=r$. By construction we then have that f is one-to-one on V with image U and so f has an inverse $g:U\to V$ where $g(w)=z$ if and only if $f(z)=w$.

We will first show that g is continuous. To this end fix a point $w_0 \in U$, $z_0 = g(w_0)$, and an $\epsilon > 0$. We may suppose without loss of generality that ϵ is small enough so that $D_\epsilon(g(w_0)) = D_\epsilon(z_0) \subseteq V$. Applying Theorem 5.7 now to the point z_0, there exists an $\eta_0 > 0$ such that for all $0 < \eta \leq \eta_0$, there is some $\rho = \rho(\eta) > 0$ such that every value in $\{|w - w_0| < \rho(\eta)\}$ is taken once in $\{|z - z_0| < \eta\}$. For η_0 as above, we may without loss of generality assume that $\eta_0 < \epsilon$ and we let δ be such that

$$\delta \leq \rho(\eta_0), \quad D_\delta(w_0) \subseteq U.$$

For $|w - w_0| < \delta$ there exists a unique $z \in V$ such that $f(z) = w$ but necessarily $|z - z_0| < \eta \leq \eta_0 < \epsilon$. Thus $g(w) = z$ and

$$|g(w) - g(w_0)| = |z - z_0| < \epsilon,$$

proving continuity.

We now conclude by showing analyticity of g. Let $w_0 \in U$ and for w nearby w_0 we set $z_0 = g(w_0), z = g(w)$. By continuity of g as $w \to w_0, z \to z_0$ and so

$$\lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \frac{1}{f'(z_0)},$$

since r was originally chosen to be small enough so that $f'(z_0) \neq 0$. □

The Inverse Function Theorem then gives for any analytic function on a domain D, for $a \in D$ such that $f'(a) \neq 0$, f is locally a homeomorphism onto its image. Consequently f is an open mapping nearby any point such that $f'(a) \neq 0$. More generally any analytic function on a domain Ω is an open map as was shown on previous homework.

Recall that a map $f : \Omega \to \mathbb{C}$ is said to be *conformal* if it preserves angles and this definition is equivalent to f be analytic and with nowhere vanishing derivative on Ω. With the theory developed now we can answer the following question: What is the most general conformal mapping of the unit disc onto itself? Suppose f is conformal such that $\{|z| < 1\}$ maps onto $\{|w| < 1\}$. Then as we have remarked it must hold that $f' \neq 0$ everywhere inside the unit disc and so by the inverser function theorem, f has a local inverse around every point in the disc which patches together to a analytic inverse f^{-1} on all of the unit disc. Being onto there is some a in the disc such that $f(a) = 0$. Consider then the Möbius transformation

$$g(z) := \frac{z - a}{1 - z/\overline{a}},$$

with the convention that if $a = 0$ we simply drop the $z/\overline{a}$ term above. Being a Möbius transformation, g is conformal everywhere and so is its inverse. Moreover g maps the unit disc onto itself. With this in mind we define two new maps,

$$h := f \circ g^{-1}, \quad k := g \circ f^{-1}.$$

Both h, k conformally map the disc onto itself and both fix the origin. Hence we may apply Schwarz Lemma—Lemma 3.11—to both h, k to see that for $|z| < 1$ both

$$|h(z)| \leq |z|, \quad |k(z)| \leq |z|.$$

Now let $z = h(w)$ for $|w| < 1$ and observe,

$$|w| = |gg^{-1}(w)| = |h(k(w))| \leq |k(w)| \leq |w|,$$

giving equality everywhere above and so $|k(w)| = |w|$. Thus by the second statement of Schwarz Lemma we conclude that $k(w) = \lambda w$ for some $|\lambda| = 1$. We then see

$$\lambda w = k(w) = g(f^{-1}(w)),$$

and so

$$f(z) = e^{i\theta} g(z) = e^{i\theta} \frac{z - a}{1 - z/\overline{a}},$$

for some θ.

Theorem 5.9. *Let C be a simple closed contour, f analytic inside and on C. Suppose that as z traces out C, $f(z)$ traces out the simple closed contour Γ. Then f maps the interior of C conformally onto the interior of Γ.*

Proof. Let w_1 be an arbitrary point inside Γ and note since Γ is simple closed that

$$\frac{1}{2\pi} \Delta_\Gamma \operatorname{Arg}(w - w_1) = 1.$$

Since f traces out Γ this is also given by

$$1 = \frac{1}{2\pi i} \Delta_C \operatorname{Arg}(f(z) - w_1) = N - P,$$

where N denotes the number of zeroes for $f - w_1$ inside C and P the number of poles. By the assumption that f is analytic, $P = 0$ and hence there is a unique solution z_1 inside C such that $f(z_1) = w_1$. Hence f is one-to-one and onto the interior of Γ inside C. Of course this must imply that $f' \neq 0$ inside C since otherwise, there would be a point where locally f is n to 1 for $n > 1$ nearby this point by Theorem 5.7.

The only thing that remains to be shown is that f maps all points inside C to the interior of Γ. To this end, let z_0 be inside C and set $w_0 = f(z_0)$. Of course

$$\frac{1}{2\pi} \Delta_C \operatorname{Arg}(f(z) - f(z_0)) \geq 1,$$

since $f - f(z_0)$ is analytic inside C and has at least one solution. The above is also equal to

$$\frac{1}{2\pi} \Delta_\Gamma \operatorname{Arg}(w - w_0),$$

and so the above is at least 1. Note however that if w_0 lies outside of Γ then the above would be 0 and thus w_0 must be interior to Γ. □

5.4. **Summation of Series Using Residues.** In this small section we show how the Residue theorem applied to very particular functions can be used to sum series, just as the theorem could be used to evaluate real integrals. We start with the following observation: If φ is a meromorphic function on all of $\mathbb{C}$ with simple poles at each integer such that

$$\operatorname{Res}(\varphi; k) = 1,$$

for every $k \in \mathbb{Z}$, then for f analytic at the integers, C a simple closed contour,

$$\frac{1}{2\pi i} \int_C f(z)\varphi(z) dz = \sum_{\substack{m \in \mathbb{Z} \\ m \text{ inside } C}} f(z) + \sigma,$$

where σ denotes the sum of the residues for f inside C at its poles. This is because at any integer $m \in \mathbb{Z}$ inside $\mathbb{C}$ we can calculate the residue for f as

$$\operatorname{Res}(f\varphi; m) = \lim_{z\to m}(z-m)f\varphi = \lim_{z\to m} f \lim_{z\to m}(z-m)\varphi = f(m).$$

Thus we aim to construct such a φ. The functions $\pi\cos(\pi z), \pi\sin(\pi z)$ both have period 2, but we claim the function

$$\varphi(z) = \pi\cot(\pi z) = \pi\frac{\cos(\pi z)}{\sin(\pi z)},$$

has period 1. Indeed by appealing to the angle sum formulas,

$$\frac{\cos(\pi(z+1))}{\sin(\pi(z+1))} = \frac{\cos\pi z\cos\pi - \sin\pi z\sin\pi}{\sin\pi z\cos\pi + \cos\pi z\sin\pi} = \frac{\cos\pi z}{\sin\pi z}.$$

Since $\sin\pi z$ has a simple zero at each integer and $\cos\pi z$ does not vanish at each integer we obtain that φ has a simple pole at each integer. Moreover since $\sin\pi z$ and $\cos\pi z$ are entire, these are the only poles for φ. We simply compute its residue at each integer by computing its residue at 0 since by periodicity this will be its residue at all integers.

$$\operatorname{Res}(\varphi; 0) = \lim_{z\to 0}\pi\frac{\cos\pi z}{\sin\pi z}\lim_{z\to 0}\cos\pi z\lim_{z\to 0}\frac{\pi z}{\sin\pi z} = 1.$$

A similar function is $\varphi(z) = \pi\csc(\pi z)$ which by similar calculations is seen to have period 2 and more generally satisfy

$$\varphi(z+k) = (-1)^k\varphi(z),$$

for any $k \in \mathbb{Z}$. Simlar calculations show that it has only simple poles at the integers and the residue at $m \in \mathbb{Z}$ is given by $(-1)^m$. This φ will be useful for summing alternating series.

Before computing some series it will be useful to make the following observation: The two function $\pi\cot\pi z$ and $\pi\csc\pi z$ are bounded outside the discs of radius δ about the integers for any $0 < \delta < 1/2$. We will show this only for $\pi\cot\pi z$ as the proof for the other function is nearly identical. By periodicity we can prove this by only showing this holds for $\pi\cot\pi z$ in the region

$$\{z| -1/2 \le \operatorname{Re}(z) \le 1/2\} \setminus D_\delta(0).$$

Of course φ is bounded on any bounded portion of this domain since φ is continuous. Thus we ask what happens as $\operatorname{Im}(z) \to \pm\infty$. Note,

$$\begin{aligned}
\pi\cot\pi z &= \pi\frac{\cos\pi z}{\sin\pi z},\\
&= \pi i\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}},\\
&= \pi i\frac{e^{\pi ix-\pi y} + e^{-\pi ix+\pi y}}{e^{\pi ix-\pi y} - e^{-\pi ix+\pi y}},\\
&= \pi i\frac{e^{2\pi ix-2\pi y} + 1}{e^{2\pi ix-2\pi y} - 1},\\
&= \pi i\frac{e^{2\pi ix}e^{2\pi y} + 1}{e^{2\pi ix}e^{2\pi y} - 1}.
\end{aligned}$$

Since $e^{2\pi i x}$ is bounded and $e^{-2\pi y}$ tends to $0, \infty$ as $y \to \pm\infty$ respectively we find that,

$$\lim_{\substack{\operatorname{Im}(z)\to\infty \\ -1/2\le \operatorname{Re}(z)\le 1/2}} \pi\cot\pi z = -\pi i, \qquad \lim_{\substack{\operatorname{Im}(z)\to-\infty \\ -1/2\le \operatorname{Re}(z)\le 1/2}} \pi\cot\pi z = \pi i.$$

We now proceed with a few examples to of computations of series using these functions. Consider first the series given by

$$\sum_{m=-\infty}^{\infty} \frac{1}{(a+m)^2},$$

for $a \notin \mathbb{Z}$. Set $f(z) = \frac{1}{(a+m)^2}$ and consider the contour integral,

$$\frac{1}{2\pi i}\int_{C_r} f(z)\pi\cot\pi z dz,$$

where C_r is the circle of radius $r + 1/2$ centered at the origin. Since f has only a single pole of order 2 at $z = -a$, we have via the Residue theorem,

$$\text{(3)} \qquad \frac{1}{2\pi i}\int_{C_r} f(z)\pi\cot\pi z dz = \sum_{m=-r}^{r}\frac{1}{(a+m)^2} + \operatorname{Res}(f\pi\cot\pi z; -a).$$

By our previous remarks, $\pi\cot\pi z$ is bounded on C_r so let us fix a bound $|\pi\cot\pi z| \le M$ on any C_r. Moreover for any z,

$$|f(z)| \le \frac{1}{(|z|-|a|)^2}.$$

Hence we may estimate the integral as follows,

$$\left|\frac{1}{2\pi i}\int_{C_r} f(z)\pi\cot\pi z dz\right| \le \frac{1}{2\pi}(2\pi(r+1/2)M\frac{1}{(r+1/2-|a|)^2},$$
$$= \frac{M(r+1/2)}{(r+1/2-|a|)^2},$$

and since as $r \to \infty$ our upper bound tends to zero we conclude that the integral tends to 0 as $r \to \infty$. In particular since $\sum_{m=-\infty}^{\infty}\frac{1}{(a+m)^2}$ converges absolutely, we may evaluate its limit via any ordering of the terms so in particular we conclude that upon taking $r \to \infty$ in Equation 3,

$$-\operatorname{Res}(f\pi\cot\pi z; -a) = \sum_{m=-\infty}^{\infty}\frac{1}{(a+m)^2}.$$

To evaluate the residue, let us expand the Taylor series for $\pi\cot\pi z$, at $-a$,

$$\pi\cot\pi z = \pi\cot -\pi a + \left[\frac{d}{dz}\pi\cot\pi z\bigg|_{z=-a}\right](z+a) + \dots,$$

to see,

$$f(z)\pi\cot\pi z = \frac{\pi\cot -\pi a}{(z+a)^2} + \left[\frac{d}{dz}\pi\cot\pi z\bigg|_{z=-a}\right]\frac{1}{(z+a)} + \dots$$

Hence,

$$\operatorname{Res}(f\pi\cot\pi z; -a) = \frac{d}{dz}\pi\cot\pi z\bigg|_{z=-a} = -\pi^2\csc^2(\pi a).$$

Using this we conclude that,

$$\sum_{m=-\infty}^{\infty} \frac{1}{(a+m)^2} = \pi^2 \csc^2(\pi a).$$

Siimilar techniques can be used to evaluate,

$$\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m^2}.$$

Using the same contours as previous, we consider the contour integral and evaluate it using the residue theorem,

$$\frac{1}{2\pi i}\int_{C_r} \frac{1}{z^2}\pi \cot \pi z dz = \sum_{\substack{m=-r \\ m\neq 0}}^{r} \frac{1}{m^2} + \mathrm{Res}\left(\frac{\pi \cot \pi z}{z^2}; 0\right).$$

Just as before we will notice that the integral tends to 0 as $r \to \infty$ from which we conclude,

$$\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m^2} = -\mathrm{Res}\left(\frac{\pi \cot \pi z}{z^2}; 0\right).$$

Now if

$$\frac{1}{z} + a + bz + \dots,$$

denotes the Laurent expansion for $\pi \cot \pi z$ about $z = 0$, it will follow that Res $\left(\frac{\pi \cot \pi z}{z^2}; 0\right) = b$. Of course, Writing out the taylor series for $\cos \pi z, \sin \pi z$ about 0 we see

$$\cot \pi z = \frac{1 - \frac{1}{2}(\pi z)^2 + \frac{1}{4!}(\pi z)^4 - \dots}{\pi z - \frac{1}{6}(\pi z)^3 + \frac{1}{5!}(\pi z)^5 - \dots} = \frac{1}{\pi z} - \frac{1}{3}\pi z + \dots$$

and hence $b = \frac{-\pi^2}{3}$. Thus we conclude,

$$\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{3},$$

or more familiarly,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

For one final calculation we try an alternating series,

$$\sum_{m=-\infty}^{\infty} \frac{(-1)^m}{(a+m)^2},$$

for some $a \in \mathbb{C} \setminus \mathbb{Z}$. Using the same contours as before, the residue theorem gives

$$\frac{1}{2\pi i}\int_{C_r} \frac{\pi \csc \pi z}{(a+z)^2} dz = \sum_{m=-r}^{r} \frac{(-1)^m}{(a+m)^2} + \mathrm{Res}\left(\frac{\pi \csc \pi z}{(a+z)^2}; -a\right).$$

Similar estimates yield,

$$\frac{1}{2\pi i}\int_{C_r} \frac{\pi \csc \pi z}{(a+z)^2} dz \to 0,$$

as $r \to \infty$ and therefore

$$\sum_{m=-\infty}^{\infty} \frac{(-1)^m}{(a+m)^2} = -\text{Res}\left(\frac{\pi \csc \pi z}{(a+z)^2}; -a\right),$$

and a nearly identical calculation as the one in the first example shows that this residue is $-\pi^2 \csc \pi a$.

6. Conformal Mappings

6.1. **Equicontinuity and Normal Families.** A familiy of functions $\mathscr{F}$, defined on some $S \subseteq \mathbb{C}$ is said to be *equicontinuous* if given any $\epsilon > 0$, there exists a $\delta > 0$ such that for any $f \in \mathscr{F}$ and any $z_1, z_2 \in S$ with $|z_1 - z_2| < \delta$ one has $|f(z_1) - f(z_2)| < \epsilon$.

Theorem 6.1. *Suppose $\mathscr{F}$ is a family of analytic functions on some disc $D_R(a)$ which are uniformly bounded on this disc. Then $\mathscr{F}$ is equicontintuous on any closed subdisc, $E := \overline{D}_{R_0}(a)$ with $R_0 < R$.*

Proof. Let M be such that $|f(z)| \leq M$ for any $f \in \mathscr{F}$ and $z \in D_R(a)$. For any $z \in E, f \in \mathscr{F}$, we may write

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2} dw,$$

where C is the circle of radius $R' - R_0$ centered at z where R' is any value such that $R_0 < R' < R$. Estimating this integral,

$$|f'(z)| \leq \frac{2\pi(R' - R_0)M}{2\pi(R' - R_0)^2} = \frac{M}{R' - R_0}.$$

Taking $R' \to R$ we see that

$$|f'(z)| \leq \frac{M}{R - R_0}.$$

Thus f' is uniformly bounded on E . Using this, note that for any $z_1, z_2 \in E, f \in \mathscr{F}$,

$$f(z_2) - f(z_1) = \int_{z_1}^{z_2} f'(z) dz,$$

where the integral is over the line segment connecting the two points. Hence

$$|f(z_2) - f(z_1)| \leq \frac{M}{R - R_0}|z_2 - z_1|,$$

yielding equicontinuity. □

Corollary 6.2. *If $\mathscr{F}$ is a familiy of analytic functions on a domain D and are uniformly bounded on D, then $\mathscr{F}$ is equicontinuous on any compact subset of D.*

Proof. If $K \subseteq D$ is compact then K has positive distance from $\mathbb{C} \setminus D$, say $\delta > 0$. We can cover K via the discs,

$$K \subseteq \bigcup_{z \in K} D_{\delta/4}(z),$$

and obtain a finte subcover $D_{\delta/4}(z_1), \ldots, D_{\delta/4}(z_n)$, since K is compact. Theorem 6.1 applies on each $D_{\delta/2}(z_j)$ and so $\mathscr{F}$ is equicontinuous on each $\overline{D}_{\delta/4}(z_j)$ from which we conclude it is equicontinuous on K. □

Theorem 6.3. *Suppose $\{f_k\}_1^\infty$ is a sequence of functions equicontinuous on a compact set $K \subseteq \mathbb{C}$ and suppose further that $\{f_k(z)\}_1^\infty$ converges for all z in some countable dense subset of K. Then f_k converges uniformly on K.*

Proof. Let $\epsilon > 0$ and let $\delta := \delta(\epsilon/3) > 0$ be as in the definition of equicontinuity corresponding to $\epsilon/3$. Further let $\{z_j\}_1^\infty$ be an enumeration of the countable dense subset of K on which the f_k converge pointwise. We can cover K via discs,

$$K \subseteq \bigcup_{j=1}^{\infty} D_\delta(z_j),$$

and by compactness about a finite subcover,

$$K \subseteq \bigcup_{j=1}^{k} D_\delta(z_j).$$

Now since the f_n converge pointwise at each of the z_j, we have positive integers $N_1, \ldots, N_k$ such that if $n \geq N_j$ one has

$$|f_{n+m}(z_j) - f_n(z_j)| < \epsilon/3,$$

for any $m > 0$. For arbitrary $z \in K$, there is a z_j with $j \in \{1, \ldots, k\}$ such that $|z - z_j| < \delta$, and thus for this z_j,

$$|f_{n+m}(z) - f_n(z)| \leq |f_{n+m}(z) - f_{n+m}(z_j)| + |f_n(z_j) - f_{n+m}(z_j)| + |f_n(z_j) - f_n(z)|.$$

By equicontinuity, the two outer terms above are at most $\epsilon/3$. Moreover if we take $n > N_j$ the middle term is also bounded above by $\epsilon/3$. Since our choice of z was arbitrary, the $\{f_n\}$ are seen to be uniformly Cauchy on K which by compactness implies uniform convergence on K. □

We say a family of functions $\mathscr{F}$ defined on a domain D is *normal* if whenever $\{f_n\} \subseteq \mathscr{F}$ there is a subsequence $\{f_{n_k}\}$ which converges uniformly on compact subsets of D.

Theorem 6.4 (Montel's Theorem)**.** *A family $\mathscr{F}$ of analytic and uniformly bounded functions on a domain D is normal.*

Proof. Let $\{z_j\}_1^\infty$ be a countable dense subset of D. Note that any compact subset $K \subseteq D$ is contained in a compact subset of D which is the closure of any open subset of D. Indeed there is some positive distance between K and $\mathbb{C} \setminus D$ and so K is contained in a finite union of discs around points in D whose distance from the boundary of D is positive. Let U denote this open cover, which has closure L lying completely in D. Since $U \subseteq D$ there is a subset $\{w_j\}_1^\infty \subseteq \{z_j\}_1^\infty$ which is dense in$U$ and so dense in L. From this we can conclude that if we can show that for any sequence $\{f_n\}$ is a subsequence converging uniformly on all compact sets of the form L then $\mathscr{F}$ will be normal.

Now since $\mathscr{F}$ are analytic and uniformly bounded on D—and so on L—if we can just for any sequence $\{f_n\}_1^\infty$ there is a subsequence f_{n_k} converging pointwise on the countable dense set $\{w_j\}_1^\infty$ we will be able to conclude via Corollary 6.2 and Theorem 6.3 that $\mathscr{F}$ is a normal family. Note that $\{f_n(w_1)\}$ is bounded and so there exists a subsequence $\{f_{1,n}\}$ so that $\{f_{1,n}(w_1)\}$ converges. $\{f_{1,n}(w_2)\}$ is bounded so there exists a further subsequence $\{f_{2,n}\}$ which converges at w_1 and w_2. Continuing indefinitely we can always find a subsequene $\{f_{k,n}\}$ of $\{f_{k-1,n}\}$, that converge at $w_1, \dots, w_k$. The diagonal sequence, $\{f_{n,n}\}$ is eventually a subsequence of each $\{f_{k,n}\}$ and so converges at every w_k. Hence the diagonal sequence, which is equicontinuous since L is compact, converges uniformly on L by Theorem 6.3. Thus $\mathscr{F}$ is normal. □

It turns out the a statement close to the converse of Montel's Theorem is also true.

Theorem 6.5. *If $\mathscr{F}$ is a normal family of analytic functions on a domain D, then $\mathscr{F}$ is uniformly bounded on every compact subset of D.*

Proof. As in the argument for Theorem 6.5, it suffices to prove this for compact sets K such that $K = \overline{U}$ for some open set U. On such K each $f \in \mathscr{F}$ is continuous and therefore bounded, say

$$\sup_{z \in K} |f(z)| = M(f).$$

Suppose by way of contradiction the set $T = \{M(f) : f \in \mathscr{F}\}$ was not bounded. Then for each n, there exists an $f_n \in \mathscr{F}$ such that $M(f_n) \geq n$. By normality, there is a subsequence $\{f_{n_k}\}$ which converges uniformly on K. The limiting function f is then analytic on K and so bounded, say $|f(z)| \leq M$. But for large enough k, $z \in K$,

$$|f_{n_k}(z)| \leq |f_{n_k}(z) - f(z)| + |f(z)| \leq 1 + M,$$

which is contradiction at the first k such that $M(f_{n_k}) \geq 1 + M$. □

Upon noting that the original argument in the proof of Montel's theorem would have held more generally if each $\mathscr{F}$ was only bounded uniformly on every compact sets, the previous two theorems combine to show a family of analytic functions $\mathscr{F}$ is normal if and only if it is uniformly bounded on every compact subset of D.

Theorem 6.6. *Let $\mathscr{F}$ be a normal family on a domain D and let $\{f_n\}$ be a sequence from $\mathscr{F}$ which converges on a set of points having a limit point in D. Then $\{f_n(z)\}$ converges at every $z \in D$ and uniformly on compact subsets of D.*

Proof. Suppose by way of contradiction that $\{f_n\}$ is such a sequence converging on a set with a limit point and $c \in D$ is such that $\{f_n(c)\}_1^\infty$ does not converge. The singleton $\{c\}$ is a compact subset of D and so since $\mathscr{F}$ is normal $\{f_n(c)\}$ is abounded sequence. As it does not converge, this implies $\{f_n(c)\}$ must have at least two distinct limit points, say a and b. Fix subsequences so that $f_{n_k}(c) \to a$, $f_{m_k}(c) \to b$. By normality, we can extract a further subsequence $\{f_{n_{k_p}}\}$ converging uniformly on compact sets to an analytic function g. Similar we have a subsequence $\{f_{m_{k_p}}\}$ converging uniformly on compact sets to an analytic function h. Necessarily $g(c) = a$ and $h(c) = b$.

Let S denote the set of points in D on which the original sequence converges pointwise. For $z \in S$, since the two subsequences above converge on all compact sets, we must therefore have that $g(s) = h(s)$, or equivalently $g - h \equiv 0$ on S. By hypothesis S has a limit point and since g, h both analytic it follows that $g - h \equiv 0$ on D. Hence $a = g(c) = h(c) = b$ contradicting the assumption that a, b were two limit points of $\{f_n(c)\}$. Thus we have shown via contradiction that $\{f_n(z)\}$ converges pointwise in all of D.

For the final statement, $\mathscr{F}$ being normal implies that $\mathscr{F}$ is uniformly bounded on all compact subsets and hence by Corollary 6.2 $\mathscr{F}$ is equicontinuous on all compact subsets. In those compact subsets, by Theorem 6.3 we have that pointwise convergence implies uniform convergence. □

6.2. **The Riemann Mapping Theorem.**

The tools of the previous section, while interesting in their own right were developed to prove the following amazing theorem.

Theorem 6.7 (The Riemann Mapping Theorem)**.** *A simply connected domain Ω on the Riemann Sphere with at least two boundary points admits a conformal mapping $F : \Omega \to D_1(0)$ which is both one-to-one and onto. Moreover if one fixes a $z_0 \in \Omega$ so that $F(z_0) = 0$ then F is unique.*

Before embarking on the proof of the Riemann Mapping Theorem let us first perform a sanity check and remark that the condition that D having at least two boundary points in necessary. For instance $\mathbb{C}$ has a single boundary point on the sphere, namely ∞, and a conformal mapping into the disc would contradict Liouville's Theorem.

Proof. We will first show that Ω is conformally equivalent to an open subset of the unit disc containing the origin. Indeed let a be some point not belonging to Ω so that $z - a$ does not vanish on the simply connected set Ω. Simply being translation, the image of Ω under $z - a$ is simply connected and does not contain the origin. Thus we can define $f(z) = \log(z - a)$ on Ω. Note that $e^{f(z)} = z - a$, and so f is injective. Pick some $w \in \Omega$ and note for any $z \in \Omega$,

$$f(z) \neq f(w) + 2\pi i.$$

This is because upon exponentiating we'd find $z = w$, an obvious contradiction. More is true: there is a disc centered at $f(w) + 2\pi i$ so that this disc contains no points of $f(\Omega)$. Otherwise there would exist a sequence of points $\{z_n\}$ in Ω such that $f(z_n) \to f(w) + 2\pi i$. Exponentiating this relation we'd find $f(z_n) \to f(w)$, again a contradiction. With all of this setup, the map

$$F(z) = \frac{1}{f(z) - (f(w) + 2\pi i)},$$

is bounded and injective. Moreover we claim it is conformal, as

$$F'(z) = \frac{a - z}{(f(z) - (f(w) + 2\pi i))^2},$$

which is never 0 in Ω. Hence F is conformal equivalence between Ω and $F(\Omega)$ and after rescaling and translating $F(\Omega)$ we may assume that $F(\Omega)$ is an open subset of the unit disc which contains the origin.

Now the previous paragraph allows to assume that our original Ω was an open subset of the unit disc containing the origin. Consider the family,

$$\mathscr{F} = \{f : \Omega \to D_1(0) \text{ analytic, injective, } f(0) = 0\}.$$

$\mathscr{F} \neq \varnothing$ as it contain the identity map and is trivially uniformly bounded. Since all of the f are uniformly bounded, it follows that the quantities $|f'(0)|$ are uniformly bounded for $f \in \mathscr{F}$, since

$$f'(0) = \int_C \frac{f(w)}{(w - z)^2} dw,$$

for some small enough circle C centered at 0. Thus we set,

$$s := \sup_{f \in \mathscr{F}} |f'(0)|,$$

and choose a sequence $\{f_n\} \subseteq \mathscr{F}$ such that $|f_n'(0)| \to s$ as $n \to \infty$. By Montel's Theorem, $\mathscr{F}$ is a normal family and hence the sequence $\{f_n\}$ admits a subsequence $\{f_{n_k}\}$ which converges uniformly on all compact subsets of Ω to some analytic function f. Note that f is not constant, since in particular $f'(0) > 1$. We claim then that f is injective. To see this let $z_0 \in D$ be arbitrary and set $w_0 = f(z_0)$. Then for any $z \neq z_0 \in D$, we may choose $\delta > 0$ such that $z_0 \in \overline{D_\delta(z)}$. For each n, define $w_n = f_n(z_0)$. Since each f_n is injective, the functions $f_n - w_n$ have no zeroes on the compact $\overline{D_\delta(z)}$ and converge to $f - w_0$. Hence by Hurwicz' Theorem—Theorem 5.6—$f - w_0$ has no zeroes in $\overline{D_\delta(z)}$ and so f is injective. By continuity we have that $|f(z)| \leq 1$ for every $z \in \Omega$ and by the Maximum Modulus Principle in fact $|f(z)| < 1$ for all $z \in \Omega$. From this we conclude that $f \in \mathscr{F}$ and f achieves the supremum, $|f'(0)| = s$.

Our claim now is that f is the desired conformal mapping onto the unit disc. We already have that f injectively into the unit disc and is analytic. Being injective, by Theorem 5.7 it must follow that $f' \neq 0$ in Ω since otherwise f would fail to be locally injective near a point where the derivative vanishses. Hence we need only show that f is surjective to conclude it is the desired conformal equivalence between Ω and the unit disc. To this end, suppose by way of contradiction there was an $\alpha \in D_1(0)$ such that $f(z) \neq \alpha$ for any $z \in \Omega$. The automorphism of the disc,

$$\psi_\alpha = \frac{\alpha - z}{1 - \overline{\alpha} z},$$

exchanges 0 and α, Now we notice that the map $\psi_\alpha \circ f$ is analytic on Ω and nowhere vanishing. Since Ω is simply connected, by Lemma 2.16 there is a map $\sigma : \Omega \to D_1(0)$ such tha $\sigma^2 = \psi_\alpha \circ f$. Consider,

$$\tau(z) = \frac{\sigma(z) - \sigma(\alpha)}{1 - \overline{\sigma(\alpha)}\sigma(z)}.$$

τ map U into the unit disc, is injective and maps 0 to 0. Hence $\tau \in \mathscr{F}$. It can then be shown,

$$|\tau'(0)| > |f'(0)|,$$

reaching the deisred contradiction. □